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SOLUTIONS OF EXAMPLES

IN

CONIC SECTIONS,

TREATED GEOMETRICALLY

BY

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THIRD EDITION, REVISED.

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PREFACE TO SOLUTIONS.

I HAVE frequently received requests for a book of Solutions of the Examples in my treatise on Conic Sections, but have never been able to find time to prepare them.

Mr Archer Green, B.A., Scholar of Christ's College, volunteered to undertake the task, with the aid of my notes and his own, and, with the exception of a few at the end, wrote out the solutions entirely.

Mr Green was however prevented by illness from completing the revision of the proofs, and I am much indebted to Mr J. Greaves, Fellow of Christ's College, who kindly undertook to examine the rest of the sheets.

The book will, I hope, prove useful both to students and teachers, as a companion volume to the treatise on Conic Sections.

W. H. BESANT.

Sept. 1881.

PREFACE TO THE THIRD EDITION.

THE solutions have been revised, and many additions have been made to them. They will now be found to be in complete accordance with the sixth edition of the Geometrical Conics.

W. H. BESANT.

Jan. 1890.

CONIC SECTIONS.

SOLUTIONS OF EXAMPLES.

CHAPTER I.

1. IF the tangent at P meet the directrix in Z, and S be the focus, PSZ is a right angle;

$\therefore$ S lies on the circle of which PZ is diameter.

2. Let PN and QM be the ordinates at P and Q.

Then $$PN : QM :: SP : SQ :: XN : XM;$$
$\therefore$ the triangles PXN and QXM are similar and PX, QX equally inclined to XS.

3. By Art. 8, FS is the external bisector of the angle PSQ.

4. $$SP : PK :: SA : AX :: SE : EK;$$
$$\therefore EP \text{ bisects the angle } SPK.$$

5. Since F, S, P and K lie on a circle,
the angle KSF = the angle FPK = the angle FTS.

6. $$PN : P'N' :: SP : SP';$$
$$\therefore XK : XN :: XK' : XN';$$
$\therefore$ the angle LNN' = the angle $K'N'X$ = the angle $LN'N$.

7. Let Q be the point where the tangent at R meets NP.

Then $NQ : NX :: SR : SX :: SA : AX :: SP : NX$;

$$\therefore SP = QN.$$

8. Let SY be perpendicular to the tangent at P and GL perpendicular to SP.

Then, since the triangles PSY, GPL are similar,

$$PG : PL :: SP : SY,$$

or $$PG : SR :: SP : SY.$$

9. If the tangent meet the directrix in Z, and SP be drawn such that ZSP is a right angle meeting the tangent in P,

then P will be the point of contact of the tangent ZP.

10. If P, Q be the extremities of the chord, and PK, QL be perpendicular to the directrix,

$$SP : PK :: SA : AX :: SQ : QL;$$

$$\therefore SP + SQ :: PK + QL :: SA : AX.$$

Now the distance of the middle point of PQ from the directrix is equal to half $PK + QL$, and is therefore least when $SP + SQ$ is least, that is, when PQ goes through the focus.

11. If TP, TP' be the fixed tangents, and the tangent at Q meet them in E, E',

the angle $PSE =$ the angle ESQ, and the angle $QSE' =$ the angle $E'SP'$;

$$\therefore \text{the angle } ESE' = \text{half the angle } PSP'.$$

12. If perpendiculars from the given points PK, QL be drawn to the directrix and S be the focus,

$$SP : SQ :: PK : QL, \text{ a constant ratio;}$$

$$\therefore \text{the locus of } S \text{ is a circle.}$$

13. Let the normal at P meet the axis in G. Taking O as the fixed point in the axis, it is obvious that the triangles OSR, GSP are similar;

$$\therefore SO : SR :: SG : SP :: SA : AX;$$

$\therefore$ SR is constant, and R lies on a circle of which S is the centre.

14. $$AT : AX :: SR : SX :: SA : AX;$$
$$\therefore AT = AS.$$

15. ST bisects the angle between SP and SQ, Art. 12, and SR bisects the angle between QS, and SP produced, Prop. II., Art. 5;

$$\therefore RST \text{ is a right angle.}$$

16. The triangles EAT, ERS are similar;

$$\therefore AT : SR :: EA : ER :: AX : SX;$$
$$\therefore AT : AX :: SR : SX :: SA : AX;$$
$$\therefore AT = AS.$$

17. If TL be perpendicular to the directrix,

$$SR : TL :: SA : AX :: SM : TL;$$
$$\therefore SM = SR.$$

18. FS is the external bisector of the angle QSP, and $F'S$ of QSP';

$\therefore$ the angle FSF' = half the angle PSP'.

19. Since the triangles SPN, SGL are similar,

$$\therefore GL : PN :: SG : SP :: SA : AX.$$

20. If the normals PG, $P'G'$ meet in Q, and QV be drawn parallel to the axis to meet the chord in V,

$$VQ : VP :: SG : SP :: SA : AX :: SG' : SP' :: VQ : VP';$$

$\therefore$ $VP = VP'$, or V bisects PP'.

21. DS is the external bisector of the angle PSQ, and ES of pSQ;

$$\therefore DSE \text{ is a right angle.}$$

22. The semi-latus rectum is an harmonic mean between SP and SP';

$$\therefore 2SP \,.\, SP' = SR \,.\, PP'.$$

23. $PE : PL :: PQ : PG :: PV : PS :: PP' : 2SP$, see Ex. 20;

$$\therefore PE : SR :: SP' : SR;$$

$$\therefore PE = SP'.$$

Similarly $$P'E = SP.$$

24. The right-angled triangles DSQ, DSE have a common hypotenuse.

Also $$SE = SR = SQ;$$

$$\therefore \text{the angle } QSE = \text{the angle } ESP.$$

25. Let S be the focus and P and Q the given points. Through P draw a straight line PK so that SP may bear to PK the given ratio of the eccentricity.

Through Q draw a straight line QL so that $SQ : QL$ in the same ratio.

With centres P, Q and radii PK, QL respectively describe circles.

The perpendicular from S on a common tangent to these circles will be axis.

26. Let the tangents at P and Q intersect in T.

Draw TN perpendicular to directrix and TM perpendicular to SP.

Then $$SM : TN :: SA : AX.$$

But ST bears a constant ratio to SM, since angle $TSM =$ half PSQ;

$$\therefore ST \text{ bears a constant ratio to } TN.$$

27. Let T be the intersection of the tangents at P and p.

Draw TK perpendicular to Pp.

Then $TK : PL :: TP : PG$ and $TK : pl :: Tp : pg$.

Again, draw GM, gm perpendicular to SP, Sp respectively, and TN, Tn perpendicular to SP, Sp respectively.

Then $TP : PG :: TN : MP :: Tn : mp :: Tp : pg$;

$$\therefore TK : PL :: TK : pl;$$

$$\therefore PL = pl.$$

CHAPTER II.

THE PARABOLA.

1. THE distance of the centre of the circle from the fixed point is equal to its distance from the fixed straight line, and therefore its locus is a parabola of which the fixed point is focus and the fixed straight line directrix.

2. Through the vertex draw a straight line making the given angle with the axis; the tangent at the point where the diameter bisecting this chord meets the curve will be the tangent required.

Or, draw a radius vector from the focus, making twice the given angle with the axis.

3. Since $TA=AN$, $PN=2AY$; $\therefore AY^2=AS.AN$.

4. Let SY' be drawn perpendicular to the line through G parallel to the tangent.

Then in the right-angled triangles YST, $Y'SG$, $ST=SG$, and the angles YST, $Y'SG$ are equal;

$$\therefore SY=SY'.$$

5. Draw SY perpendicular to the tangent and YA perpendicular to the axis.

Produce SA to X, making AX equal to SA.

Then the straight line through X perpendicular to SX is the directrix.

6. Let the circle touch the fixed circle in Q, and the straight line in R; let P be its centre, and S the centre of the fixed circle.

Produce PR to M, making RM equal to SQ, then the

straight line MX drawn through M parallel to the given line is a fixed straight line.

Then, since SP is equal to PM, the locus of P is a parabola of which S is focus and MX directrix.

7. Draw SY, SY' perpendiculars on the two tangents.

Then, if SA be perpendicular to YY', A is the vertex.

Produce SA to X, making AX equal to AS; X is the foot of the directrix.

8. If the tangent at the end of the latus rectum meet PN in Q,

$$QN = XN = SP.$$

9. Since SYP and PNS are right angles, P, N, S, Y lie on a circle;

$$\therefore\ TY \cdot TP = TS \cdot TN.$$

10. SE is half TP,

and $$PT^2 = PN^2 + TN^2 = 4AS \cdot AN + 4AN^2;$$

$$\therefore\ SE^2 = AN \cdot XN = AN \cdot SP.$$

11. If SY be drawn perpendicular to the tangent and A be vertex, SAY is a right angle;

$\therefore$ A lies on the circle of which SY is diameter.

12. Draw SY perpendicular to the tangent, then if the circle described with centre S and radius equal to a quarter of the latus rectum meet the circle described on SY as diameter in A, A is the vertex.

Produce SA to X, making AX equal to SA, then X is the foot of the directrix.

13. $$SN : SP :: SN' : SP',$$

or $$AN - AS : AN + AS :: AS - AN' : AS + AN';$$

$$\therefore\ AN : AS :: AS : AN';$$

$$\therefore\ AN \cdot AN' = AS^2.$$

Again, $4AS \,.\, AN : 4AS^2 :: 4AS^2 : 4AS \,.\, AN'$;

$$\therefore\ PN^2 : SR^2 :: SR^2 : P'N'^2,$$

or $$PN : SR :: SR : P'N';$$

$\therefore$ the latus rectum is a mean proportional between the double ordinates.

14. Let V be the middle point of the focal chord PSP', and let the diameter through V meet the curve in Q; then, if QT, QM be the tangent and ordinate at Q, and VL be ordinate of V,

$$VL = QM \text{ and } TM = SL;$$

$$\therefore\ VL^2 = QM^2 = 4AS \,.\, AM = 2AS \,.\, TM = 2AS \,.\, SL.$$

Hence the locus of V is a parabola of which S is vertex and SL axis.

15. If P, P' be the given points, PK, $P'K'$ perpendiculars on the directrix, the focus is the point of intersection of a circle centre P, radius PK with a circle of which P' is centre and $P'K'$ radius.

In general two circles intersect in two points, therefore two parabolas can be drawn satisfying the given conditions.

16. If PG be normal at P, the triangles PNG, pPR are similar;

$$\therefore\ Pp : PN :: RP : NG;$$

$$\therefore\ RP = 2NG = \text{latus rectum};$$

$\therefore$ the locus of R is an equal parabola having its vertex A' on the opposite side of X, such that AA' is equal to the latus rectum.

17. Let P, P' be the given points, S the given focus.

A common tangent to the circles described with centres P, P' and radii PS, $P'S$ respectively will be the directrix.

18. If SP be the focal distance and SY perpendicular to the tangent at P, Y lies on the circle of which SP is diameter.

Also the angle AYS = the angle SPY;

$$\therefore AY \text{ touches the circle.}$$

19. The tangents at the ends of the focal chord PSP' meet in F on the directrix at right angles: also the straight line through F at right angles to the directrix bisects PP' in V;

$$\therefore FV = VP = VP';$$

$\therefore$ the directrix touches the circle described on PP' as diameter.

20. Draw the farther tangent to the circle parallel to the given diameter, then the locus of the point is a parabola of which the centre is focus, and the tangent thus drawn directrix.

21. Draw a straight line parallel to the given straight line, on the farther side of it, and at a distance from it equal to the radius of the circle, then the locus of the point is a parabola of which the centre of the circle is focus, and the straight line thus drawn directrix.

22. Let Q be the centre of the circle touching the sector in R and AC in M.

Through C draw CB at right angles to AC, and on the same side of it as Q, and draw QN perpendicular to the tangent at B.

Then
$$NQ + QM = BC = CQ + QR;$$
$$\therefore CQ = QN;$$

$\therefore$ Q lies on a parabola of which C is focus and BN directrix.

23. Y is the middle point of TP and Z of PG;
therefore YZ is parallel to the axis.

24. If SQ be perpendicular to the normal PG,

$$PQ = QG,$$

and if QM be the ordinate, $NM = MG$;

$$\therefore SM = AN \text{ and } PN = 2QM;$$
$$\therefore QM^2 = AS \,.\, AN = AS \,.\, SM;$$

$\therefore$ Q lies on a parabola of which S is vertex and SG axis.

25. The triangle PSG is isosceles; therefore GL is equal to PN.

26. If the circle described with centre S and radius equal to the perpendicular from S on the tangent at P meet the circle of which SP is diameter in Y, and the angle SYA be made equal to the angle SPY, then the foot of the perpendicular SA on YA will be the vertex.

27. Since SQ is double SA, ASQ (and likewise QSP) is equal to the angle of an equilateral triangle; therefore SP and SQ are equally inclined to the latus rectum.

28. $QX^2 = SX^2 + SQ^2 + 2SX.SQ$

$$= 4AS^2 + QG^2 + 2SQ.NG$$

$$= 4AS^2 + QN^2 + 2QN.NG + NG^2 + 2SQ.NG$$

$$= 4AS^2 + QN^2 + NG^2 + 2NG.SN$$

$$= 4AS^2 + NQ^2 + 2AN.NG$$

$$= 4AS^2 + QN^2 + PN^2 = 4AS^2 + QP^2.$$

29. The angle

$$SPF = SPG - FPG = SGP - GPH = SHP;$$

therefore the triangles SPF, SHP are similar;

$$\therefore\ SF.SH = SP^2 = SG^2.$$

30. A, B, C, S lie on a circle; therefore, if D be the end of the diameter drawn through S, DA, DB, DC are perpendicular to SA, SB, SC respectively.

31. Since PQ and PR are equally inclined to the axis, the circle through P, Q, R touches the parabola at P; therefore PQ is a diameter of this circle.

Therefore PRQ, the angle in a semicircle, is a right angle.

32. Let MR and AQ meet in V.

Draw the ordinates VW, RZ.

Then $MW : MZ :: WV : RZ :: AW : AN$;

$$\therefore MW : AW :: MZ : AN;$$

$$\therefore AN : AW :: AZ : AN.$$

Again, $VW^2 : QN^2 :: AW^2 : AN^2$;

$$\therefore VW^2 : RZ^2 :: AW : AZ;$$

$\therefore$ V lies on the curve.

33. Let P, Q be the given points. Bisect PQ in V, and draw VT parallel to the axis meeting the given tangent P in T.

Draw PS, QS such that TP, TQ may be equally inclined to the axis and to SP, SQ respectively. PS, QS meet in the focus.

Through P draw a straight line PK parallel to the axis, making PK equal to SP, then the straight line through K at right angles to PK will be the directrix.

34. Let P be the vertex and QVQ' be corresponding ordinate.

Take M in VP produced such that

$$QV^2 = 4MP \cdot PV.$$

Make angle TPS equal to the angle MPT, PT being parallel to QQ', and make PS equal to PM.

Then S is the focus, and the straight line drawn through M at right angles to PM is the directrix.

35. $PM^2 : QN^2 :: AM : AN$, QN being the ordinate of Q;

$$\therefore AM = 4AN \text{ and } 3AM = 4NM;$$

$$\therefore 3AT = 4QN = 2PM.$$

36. Draw PN perpendicular to AB.

Then $AN : NP :: CQ : AC :: NP : AC$;

$$\therefore PN^2 = AC \cdot AN.$$

Therefore the locus of P is a parabola of which A is vertex and AB axis.

37. The triangles LKP, PSK, KSA and TKA are similar;

$$\therefore\ KL^2 : SP^2 :: KP^2 : KS^2 :: KA^2 : AS^2 :: TA : AS$$
$$:: SP - AS : AS.$$

38. With centre S and radius one-fourth of the chord describe a circle meeting the parabola in P. The chord through S parallel to the tangent at P will be the chord required.

39. $PN^2 = 4AS . AN = 4AS . AN' + 4AS^2$
$$= P'N'^2 + N'G'^2 = P'G'^2.$$

40. If Pp, $P'p'$ be two parallel chords, and V, V' their middle points, VV' is a diameter. Let VV' meet the curve in Q.

Draw QT parallel to Pp, then QT is the tangent at Q.

Produce VV' to a point M such that $PV^2 = 4QM . QV$, then the straight line drawn through M at right angles to MV is the directrix.

Make the angle TQS equal to the angle TQM.

Then if QS be made equal to QM, S is the focus and the straight line through S perpendicular to the directrix is the axis.

41. Let the tangents at P and P' intersect in T.

Then $$4SP . PV : 4SP' . P'V'$$
$$:: P'V'^2 : PV'^2 :: TP^2 : TP'^2 :: SP : SP';$$
$$\therefore\ PV = P'V'.$$

42. If in the preceding Example $P'T$ meets PV in Z and the sides of the triangle ABC are parallel to ZP, PT and TZ respectively,

$$AB : AC :: TZ : TP :: TP' : TP.$$

43. If U, V be the vertices of the diameters bisecting Pp, Qq,

$$PS . Sp : QS . Sq :: SU : SV :: Pp : Qq.$$

44. Draw RW, LZ parallel to QQ'.

Then $$PL^2 : PR^2 :: LZ^2 : RW^2$$
$$:: QV^2 : RW^2 :: PV : PW :: PN : PR;$$
$$\therefore PL^2 = PR . PN.$$

45. This question is solved in Conics, Art. 212, p. 217.

46. $$PN^2 : AN^2 :: AM^2 : QM^2;$$
$$\therefore 4AS : AN :: AM : 4AS.$$

47. Let AP, Ap meet the latus rectum in L and l respectively.

Then $PN^2 : SL^2 :: AN^2 : AS^2 :: AN : An$, (Example 13)
$$:: PN^2 : pn^2;$$
$$\therefore SL = pn.$$
In like manner $$Sl = PN.$$

48. If PK, QL be perpendicular to the directrix, and QL' to PK produced, the angle SPQ = the angle QPL';
$$\therefore PL' = SP = PK;$$
$$\therefore SQ = QL = KL' = 2PK = 2SP.$$

49. Is equivalent to Example 32.

50. Let Q be the point of intersection, and let QK be the ordinate of Q.

Then $$AK : QK :: PN : NT;$$
$$\therefore 2AK . AN = QK . PN = PN^2 = 4AS . AN;$$
$$\therefore AK = 2AS,$$
or Q lies on a fixed straight line parallel to the directrix.

51. Let Tp, Tq be the fixed tangents, and let PQ touch the curve in R.

Then $$SP^2 = Sp . SR = Sq . SR = SQ^2;$$
$$\therefore SP = SQ.$$

52. Let TM be the ordinate of T, and TW perpendicular to SP.

$$TM^2 = ST^2 - SM^2 = TW^2 + SW^2 - SM^2$$
$$= TW^2 + XM^2 - SM^2 = TW^2 + XS^2 + 2XS \, . \, SM;$$

$\therefore$ the locus of T is a parabola of which XS is axis.

If $TW = 2AS$, $TM^2 = 4AS \, . \, XM$, or X is the vertex.

53. Let the chord PQ meet the axis in O, and the tangent at A in V.

Then by Art. 48, $VO^2 = VP \, . \, VQ$;

$\therefore$ V is a fixed point, and the locus of A is the circle of which OV is diameter.

54. Let the diameter TV meet the curve in R.

Then the tangent at R, being parallel to PQ, meets TP at right angles in Z on the directrix.

Also $$TZ : ZP :: TR : RV;$$
$$\therefore TZ = ZP.$$

Therefore T and P are equidistant from the directrix.

55. Let PT meet the axis in t.

Then $$PQ : PT :: 2PV : PT :: 2PG : Pt,$$
$$:: 2PN : Nt :: PN : AN.$$

56. If the tangents TP, TQ are equal, T lies on the axis.

Let the tangent at R meet them in p and q.

Then, since T, p, q and S lie on a circle, the triangles SqT, SpP are similar;

$$\therefore Tq : pP :: TS : SP;$$
$$\therefore Tq = pP.$$

So $$Tp = qQ.$$

57. $$AN.NL = PN^2 = 4AS.AN;$$
$$\therefore NL = 4AS.$$
But $$NG = 2AS;$$
$\therefore LG$ = half the latus rectum.

58. By Art. 5, $P'S$, $Q'S$ are the external bisectors of the angles PSA, QSA;

therefore $P'SQ'$ is a right angle.

59. The angles TCS, DRS are equal, being supplements of equal angles SCP, SRC, Art. 35.

And the angle $CTS = TQS = RDS$;

$\therefore$ the triangles TCS, DRS are similar;
$$\therefore DR : TC :: RS : SC :: RC : CP;$$
$$\therefore PC : CT :: CR : RD.$$
Similarly $$TD : DQ :: CR : RD.$$

60. Let AD and XP intersect in Q, and let QM be the ordinate.

Then $QM : DS :: AM : AS$ and $QM : PN :: XM : XN$;
$$\therefore AM : XM :: AS : XN;$$
$$\therefore AM : AS :: AS : AN;$$
$$\therefore QM^2 : PN^2 :: AM^2 : AS^2 :: AM : AN,$$
or Q is on the parabola.

61. By Example 18, YY', the tangent at the vertex, is a common tangent.
$$SY^2 = AS.SP, \qquad SY'^2 = AS.Sp;$$
$$\therefore YY'^2 = SY^2 + SY'^2 = AS.Pp.$$

62. If PV be the diameter bisecting AQ,
$$AM = 4AN.$$

Also $$AM.MR = QM^2 = 4AS.AM;$$
$$\therefore MR = 4AS.$$

Now focal chord parallel to AQ
$$= 4SP = 4XN = 4AS + AM = AR.$$

63. Let AR, CP meet in p.

Draw pN, pD perpendicular to CA, CR, and let Dp meet the tangent at A in M.

$$Cp : CP :: CD : CR :: Np : CR :: AN : AC;$$
$$\therefore Cp = AN = pM.$$

Therefore the locus of p is a parabola of which C is focus and AM directrix.

64. If QMQ' be the common chord,
$$9AS^2 = 4AQ^2 = 4AM^2 + 4QM^2 = 4AM^2 + 16AM.AS;$$
$$\therefore AM \text{ is half } AS.$$

65. Let the fixed straight line BK meet the tangent at P in K.

Draw KY' at right angles, and SY' parallel to KP.

Draw $Y'A'$ perpendicular to the axis, and KL parallel to BA.

Then, since $KY = SY'$, $SA' = KL = BA$

therefore A' is a fixed point.

Therefore KY' touches the parabola of which S is focus and A' vertex.

66. $$QD.DR = QM^2 - DM^2 = QM^2 - PN^2$$
$$= 4AS.AM - 4AS.AN = 4AS.PD.$$

67. Draw the double ordinate QMq; then, if the diameter through Q' meet Qq in D',
$$QD'.D'q = 4AS.Q'D'.$$

Now $$NT : PN :: Q'D' : QD' :: D'q : 4AS;$$

$$\therefore\ D'q : 4AS :: 2AN : PN :: PN : 2AS;$$

$$\therefore\ 2PN = D'q = D'M + Mq = QM + Q'M'.$$

Therefore the line through P bisecting QQ' is parallel to the axis.

Hence the locus of the middle points of a series of parallel chords is a straight line parallel to the axis.

68[1]. Take CP, CQ two tangents such that PCQ is two-thirds of a right angle; join SC cutting the curve in R, and draw the tangent ARB. Then, Art. 38,

$$CSP = CSQ = 120^0, \text{ and } CAR = \tfrac{1}{2}CSQ = 60^0;$$

$$\therefore\ CAB \text{ is equilateral.}$$

69. Draw AZ, AN perpendicular to the tangent and SY respectively, and draw SM perpendicular to ZA.

Then $$SM^2 = AN^2 = YN . NS = ZA . AM.$$

Therefore the locus of S is a parabola of which A is vertex and ZM axis.

70. If GZ be drawn parallel to PY and SZ to PG, then SY, SZ are equal.

Therefore, if ZB be perpendicular to the axis, $BS = AS$.

Hence GZ touches an equal parabola of which B is vertex and S focus.

71. If pqr be the triangle formed by the given straight lines, describe a parabola passing through p, q and r having its axis parallel to AS. (Ex. 45.)

If s be the focus of this parabola, draw SP parallel to sp, PQ to pq, and PR to pr.

Then $$PQ : pq :: SA : sa :: PR : pr,$$

and the angles QPR, qpr are equal.

[1] If a parabola touch the sides of an equilateral triangle, the focal distance of any vertex of the triangle passes through the point of contact of the opposite side.

72. Let RW be the ordinate of R.

Then

$$AN^2 : AW^2 :: PN^2 : RW^2 :: PN^2 : QM^2 :: AN : AM;$$

$$\therefore AN : AW :: AW : AM,$$

or

$$WN : AN :: MW : AW;$$

$$\therefore RL : QR :: AN : AW :: PN : NL.$$

73. Let PV be the ordinate to the diameter RQM.

Then

$$PM : RM :: PN : TN$$

$$:: 2PN . AS : 4AS . AN :: 2AS : PN;$$

$$\therefore PM . PN = 2AS . RM.$$

But

$$PM^2 = 4AS . QV = 4AS . RQ;$$

$$\therefore RM : RQ :: 2PN : PM :: PP' : PM;$$

$$\therefore QM : QR :: P'M : PM.$$

74. Let PP' be the chord, TWV its diameter, RQM the line parallel to the axis.

Then

$$PM : RM :: PV : TV :: PV : 2WV$$

$$:: 2VP . SW : 4SW . WV :: 2SW : PV;$$

$$\therefore PM . PV = 2SW . RM.$$

But

$$PM . MP' = 4SW . QM,$$

$$\therefore RM : QM :: 2PV : MP',$$

or

$$RQ : QM :: PM : MP'.$$

75. SR, Sr are the exterior bisectors of the angles PSQ, pSQ respectively.

Therefore RSr is a right angle.

Therefore SD, which is half the latus rectum, is a mean proportional between DR and Dr.

76. Let PVP' be parallel to the given straight line, $QV'Q'$ the chord joining the two other points of intersection of the parabola and circle.

Let the diameters through V and V' meet the curve in p and p'.

Then pp' is a double ordinate; draw $V'H$ parallel to pp' to meet pV.

VV' is perpendicular to QQ', and therefore parallel to the normal at p';

$$\therefore VV' : p'g :: V'H : p'n;$$
$$\therefore VV' = 2p'g.$$

77. The arcs QU and RV are equal, since QV and UR are parallel.

Therefore QR and UV are equally inclined to QV, that is to the axis.

But QR and the tangent at P are equally inclined to the axis;

therefore UV is parallel to the tangent at P.

78. $$VR : VR' :: VR : V'Q' :: PV : PV'$$
$$:: QV^2 : Q'V'^2 :: QV^2 : VR'^2;$$
$$\therefore VR \,.\, VR' = QV^2.$$

79. If FR, QE meet the tangent at P in V and T,

$$TE : EQ :: VR : RF :: PF : FQ. \quad \text{(Ex. 74.)}$$

Therefore EF is parallel to TP.

80. If Q be the vertex of the diameter bisecting the chord Rr which meets the diameter PW in W,

$$RW \,.\, Wr = 4SQ \,.\, PW.$$

Therefore the rectangle under the segments varies as the distance of the point of intersection W from P.

81. QS, $Q'S$ are equally inclined to SP, and therefore to the axis.

Therefore $Q'S$ meets the curve at the end of the double ordinate QMq, and, since $AM \,.\, AM = AS^2$, the semi-latus rectum is a mean proportional between QM and $Q'M'$.

Also, since the diameter through P bisects QQ', PS is an arithmetic mean between QM and $Q'M'$.

82. BB' will bisect $C'A'$ in V.
Let V' be the middle point of $B'B''$.
VV' is parallel to the axis.
And BB'' is parallel to VV', and therefore to the axis.
Similarly AA'' and CC'' are parallel to the axis.

83. Let C be the centre of the circle.

The angle between tangents to circle $=PCP'=2PSP'$ $=4$ times angle between the tangents to the parabola.

84. The tangents at the ends of the focal chord PSP' will meet in T on the directrix.

If the normals at P and P' meet in Q, TQ will be parallel to the axis.

Let TQ meet the curve in p and PP' in V. Let QM be the ordinate of Q.

Then $$XM=TQ=2TV=4Sp=4Xn.$$

Therefore, if we take B in XM such that $XB=4AS$,

$$BM=4An,\ QM^2=pn^2=4AS.An=AS.BM.$$

Hence the locus of Q is a parabola of which B is vertex and BM axis.

85. Produce PA to P', making AP' equal to AP.

On AP' as diameter describe a circle meeting the tangent at P in T.

Join TA and produce to N, making AN equal to AT.

In AN take a point S such that $PN^2=4AS.AN$, then S is focus.

86. If G be the intersection of the normals and Q vertex of the diameter bisecting PSp,

$$PS.Sp=AS.Pp=AS.TG.$$

87. If pq be a tangent parallel to PQ, $Tp=pP$, and T, p, q, S and O lie on a circle.

Therefore the angles TSO, TpO are equal, and TpO is a right angle.

88. $$SM^2 : AN^2 :: QM^2 : PN^2 :: AM : AN;$$
$$\therefore SM^2 = AM . AN.$$
So $$SM'^2 = AM' . AN;$$
$$\therefore MM' . AN = MM' . (SM - SM');$$
$$\therefore SM - SM' = AN.$$
$$MM' = SQ - SQ';$$
$$\therefore MM' : SM - SM' :: AP : AN;$$
$$\therefore MM' = AP.$$

89. If P, Q, P', Q' be the points of intersection, PQ, $P'Q'$ are equally inclined to the axis.

Hence the middle points of PQ and $P'Q'$ are equidistant from the axis.

Therefore, if P, Q be on one side of the axis and $P'Q'$ on the other, the sum of the ordinates of P and Q is equal to the sum of the ordinates of P' and Q'.

If P' be on the same side of the axis as P and Q, the ordinate of Q' is equal to the sum of the ordinates of P, Q, and P'.

90. Let the diameter through T meet the curve in W, PQ in V, and PN in t.

Let WZ be the ordinate of W; draw Qq parallel to the axis to meet PN.

$$QM . PN = PN . qN = tN^2 - Pt^2 = WZ^2 - 4AS . WV$$
$$= 4AS . AZ - 4AS . LZ = 4AS . AL.$$

91. $$pX : XA :: PN : AN :: 4AS : PN$$
$$:: 4AS . QM : 4AS . AL. \quad \text{(Ex. 90.)}$$
So $$qX : XA :: PN : AL;$$
$$\therefore pX + qX : XA :: PN + QM : AL;$$
$$\therefore pX + qX : PN + QM :: XA : AL :: tX : TL.$$
But $$NP + QM = 2TL;$$
$$\therefore pX + qX = 2tX,$$
or $$pt = tq.$$

92. Let TF, TD be drawn parallel to PE, QE normals at P and Q.

The angle $TFQ = PEQ =$ supplement of $PTQ = TSQ$;

$$\therefore\ Q, S, F, T \text{ lie on a circle.}$$

Therefore TSF is a right angle.

So TSD is a right angle, and DF goes through S.

93. If pq be a tangent parallel to PQ, $Tq = qQ$.

Also, T, p, q and S lie on a circle;

therefore the angles Tpq, TSq are equal.

Therefore TSq is a right angle.

94. Let RO be the diameter through the given point O.

Take T in OR produced such that $TR : RO$ in the given ratio.

If TP be a tangent, the chord POQ will be divided as required. (Ex. 74.)

95. If QN be the ordinate,

$$BP + PQ = QN + BX - NX = BX + QN - SQ,$$

which is greatest when $QN = SQ$, that is when Q is on the latus rectum.

96. If SZ and PG meet in Q and QT be ordinate,

$$TA : AS :: QZ : ZS :: QP : PG :: TN : NG;$$

$$\therefore\ TN = 2TA.$$

97. If QV be the ordinate of the point of contact,

$$TP = PV.$$

Therefore the distance of V from TQ is twice the distance of P, or the locus of V is a straight line parallel to TQ.

98. If $TPSQ$ be the parallelogram, the angles TSP, TSQ are equal;

therefore $TPSQ$ is a rhombus and T lies on the axis.

Therefore TSP is the angle of an equilateral triangle.

99. If SZ, SZ' be the perpendiculars on the second tangents, TQ, TQ' and PP' be the common tangent, SY perpendicular to it,

then angle $$A'SY = ASY = YSP;$$

$$\therefore A' \text{ lies in } SP, \text{ and } A \text{ in } SP';$$

$$\therefore SP = SP'.$$

Now $$SQ \, . \, SP = ST^2 = SQ' \, . \, SP';$$

$$\therefore SQ = SQ';$$

$$\therefore SZ = SZ'.$$

100. If the tangent meet AY in Y and the other parabola in Q,

$$QM^2 = \tfrac{1}{2}AS \, . \, AM, \quad AY^2 = AS \, . \, AT,$$

$$QM : AY = MT : AT;$$

$$\therefore 2TM^2 = AT \, . \, AM.$$

This can be constructed by taking $AM = MT$, or by taking $AM = 2AT$, the two solutions corresponding to the two points in which the parabola is cut by the tangent.

CHAPTER III.

THE ELLIPSE.

1. $$SD^2 = BC^2 = CS \,.\, SX.$$
Therefore CDX is a right angle.

2. ST, SP are equally inclined to PT, since pST is parallel to $S'P$.

Therefore $$ST = SP.$$

3. $$PN : PG' :: SY : SP :: BC : CD$$
$$:: PF : AC :: AC : PG'.$$

Therefore $$PN = AC.$$

4. T lies on a circle of which QQ' is a diameter and V centre;

therefore $$VT = VQ.$$

Now $$QV^2 : CP^2 - CV^2 :: CD^2 : CP^2,$$

or $$VT^2 : CV \,.\, CT - CV^2 :: CD^2 : CP^2.$$

Therefore $$TV : VC :: CD^2 : CP^2.$$

Therefore $$CT : CV :: CD^2 + CP^2 : CP^2,$$

or $$CT^2 : CV \,.\, CT :: AC^2 + BC^2 : CP^2.$$

Therefore $$CT^2 = AC^2 + BC^2.$$

5. Through T draw a straight line at right angles to AA' meeting AP, $A'P$ in E, E'.

Then $$ET : PN :: AT : AN :: CT - CA : CA - CN.$$

Now $$CT : CA :: CA : AN;$$

$$\therefore CT+CA : CT-CA :: CA+AN : CA-CN;$$

$$\therefore ET : PN :: A'T : A'N :: E'T : PN.$$

Hence PT bisects any straight line parallel to ET terminated by $A'P$, AP.

6. Draw CD parallel to the given line, and CP parallel to the tangent at D.

The tangent at P will be parallel to CD and the given line.

7. $$SR : XE :: SA : AX :: SR : SX.$$

Therefore $$XE=SX,$$

and $$AT=AS.$$

8. Draw GL perpendicular to SP.

Then $$PL=SR,$$

and $$SY : SP :: PL : PG :: SR : PG.$$

The angle SPS' is greatest when SPY is least, that is when $SY : SP$ or $BC : CD$ is least.

Hence SPS' is greatest when CD is greatest, that is when
$$CD=AC.$$

Hence SPS' is greatest when P is on the minor axis.

9. $$CE'^2=CP^2+PE'^2+2PF.PE'=CD^2+CP^2$$
$$+2CD.PF=AC^2+BC^2+2AC.BC;$$
$$\therefore CE'=AC+BC.$$

So $$CE=AC-BC.$$

$$(CP+CD)^2=AC^2+CB^2+2CP.CD,$$

which is greater than $(AC+BC)^2$, since $CP.CD$ is greater than $PF.CD$ or $AC.BC$.

Similarly $CP\sim CD$ is less than $AC-BC$.

10. Let $S'Q$ drawn parallel to SP meet the normal in K, and SY in Q.

Then $S'K=S'P$ and $KQ=SP$;

therefore $S'Q=AA'$.

11. $SY:S'Y'::YP:PY'::TP-TY:TY'-TP$
$::PG-SY:S'Y'-PG.$

12. $PS'Q$ is the supplement of $QPS'+PQS'$, and is therefore equal to the excess of twice $QPT+PQT$ over two right angles,

that is, is the supplement of twice PTQ.

13. Since CZ and SP are parallel, the angle
$$CZP=SPY=SNY;$$
therefore Y, Z, C, N lie on a circle.

14. Let AQ and SP meet in R.

Then $SA:SR::SG:SP::SA:AX$.

Therefore R lies on a circle of which S is centre.

15. Since KPt is a right angle, t lies on a circle which passes through S, P, S', K,

therefore $GK:SK::S'G:SP::SA:AX$,

and $St:tK::SY:SP::BC:CD$.

16. If SP meet $S'Y'$ in Z, then since $S'Y'=Y'Z$, SY' will bisect PG.

17. Let the circle whose centre is P touch the circles whose centres are S, H in Q, R.

Then $SP+PH=SQ+QP+PH=SQ+HR$.

Hence the locus of P is an ellipse of which S and H are foci.

18. $TN:TC::PN:Ct$.

Therefore $TN.NG:CT.NG::PN^2:Ct.PN$.

But $PN^2=TN.NG$.

Therefore $CT.NG=Ct.PN=CB^2$.

19. $TP : TQ :: CD : AC :: BC : PF :: PG : BC.$

20.
$$\begin{aligned} PN^2 : AF.A'F' &:: TN^2 : TA.TA' \\ &:: TN^2 : CT^2-CA^2 :: TN : CT \\ &:: CT-CN : CT :: CA^2-CN^2 : CA^2; \\ \therefore\ AF.A'F' &= BC^2. \end{aligned}$$

21. The perpendiculars from T on SP, SQ, HP, HQ are all equal.

Hence a circle can be described with centre T to touch
$$SP,\ SQ,\ HP,\ HQ.$$

22. If P, Q be two points of intersection, PC bisects the angle ACa and QC bisects $A'Ca$.
Therefore PCQ is a right angle.

23. If SP, HQ meet in R,
$$PSQ+PHQ=2PRQ-SQH-SPH,$$
and $SQH+SPH+2RQT+2RPT=4$ right angles,
$$\therefore\ PSQ+PHQ=\text{twice the supplement of } QTP.$$

24. Since t, P, S, g lie on a circle,
the angle $$PSt=Pgt=STP.$$

25. $Q'M : PM :: BC : AC :: PN : QN.$
Therefore $$Q'M : CN :: CM : QN.$$
Therefore QQ' passes through C.

26. $$SY : SP :: BC : CD.$$
Therefore $$SY.CD=SP.BC.$$

27. If T be intersection of tangents at A and B, then, since TC bisects AB, it is a diameter of the conic.
Therefore the tangent at C is parallel to AB.

28. The angles SPT, HPt are equal.
Also $$TP.Pt=CD^2=SP.PH,$$
or $$TP : SP :: HP : Pt.$$
Therefore SPT, HPt are similar.

29. $$PE = PE' = AC.$$

Therefore $SE = HE'$, and the angles SCE, HCE' are equal.

Therefore the circles circumscribing SCE, HCE' are equal.

30. The angles KPG, GPL are equal; therefore KL is a double ordinate of the circle of which PG is diameter.

31. If Q be the centre, QN the ordinate, and T, T' the points where the tangent at P meets the tangents at the vertices,

$$QN^2 : SN.NH :: AT.A'T' : AH.A'S :: BC^2 : A'S^2.$$
(Ex. 20.)

32. Since the tangents are equally inclined to SP, $S'P$ respectively, the bisector of the angle between them bisects SPS', and therefore passes through the point where the axis minor meets the circle.

33. If $PQRS$ be the quadrilateral, p, q, r, s points of contact, H the focus,

the angle $$pHP = PHs,\ pHQ = QHq,$$
$$SHr = SHs,\ rHR = RHq.$$

Therefore $PHQ + SHR = PHS + QHR =$ two right angles.

34. $$SG : SC :: SP :: SY \text{ (see Ex. 15)}$$
$$:: CD : BC :: PV : VA.$$

35. The normals at P and Q will meet on the minor axis in K.

Then angle between the tangents $= PKQ = PSQ$.

36. The auxiliary circle lies entirely without the ellipse except at A and A'; therefore AA' is the greatest diameter.

The circle described on BB' as diameter lies wholly within the ellipse; therefore BB' is the least diameter.

37. Let any circle passing through N and T meet the auxiliary circle in Q.

Then $$CN\,.\,CT = CA^2 = CQ^2.$$

Hence CQ touches the circle at Q, and the circle cuts the auxiliary circle orthogonally.

38. The angle $PNY = PSY = PS'Y' = PNY'$.

Therefore $$PY : PY' :: NY : NY'.$$

39. $$PQ^2 : TQ^2 :: SY\,.\,S'Y' : TY\,.\,TY'.$$

But $$TQ^2 = TY\,.\,TY'.$$

Therefore $$PQ^2 = SY\,.\,S'Y' = BC^2.$$

Therefore $$PQ = BC.$$

40. If QN and PM be the perpendiculars on the given lines passing through C, R their point of intersection,

$$RN : QN :: CP : CQ;$$

therefore the locus of R is an ellipse of which the outer circle is the auxiliary circle.

41. $$SP : S'P :: SY : S'Y' :: SY^2 : BC^2,$$

and $$S'Q : SQ :: S'Z' : SZ :: BC^2 : SZ^2.$$

Therefore $$SP\,.\,S'Q : S'P\,.\,SQ :: SY^2 : SZ^2.$$

42. Let Ca, Cb be the conjugate diameters, and Pm, Pn ordinates of P.

Then $$Cm\,.\,CM = Ca^2,$$

and $$Cn\,.\,CN = Cb^2;$$

$$\therefore CM\,.\,Pm : Ca^2 :: Cb^2 : Pn\,.\,CN;$$

$\therefore$ the triangle CPM varies inversely as the triangle CPN.

43.

$$CAV : CPT :: CA^2 : CT^2 :: CN : CT :: CPN : CPT;$$

therefore the triangles CAV, CPN are equal.

44. Let TPQ, Tpq be the tangents intersecting the auxiliary circle in P, Q, p, q.

Let E, e be their middle points.

$$PE^2+pe^2=ET^2+Te^2-TP\,.\,TQ-Tp\,.\,Tq$$
$$=CT^2-2TP\,.\,TQ=CT^2+2CA^2-2CT^2=SC^2.$$

45. Let QQ' be a diameter equally inclined to the axis with the conjugate to $P'P$.

Then the circles described through P, P', Q and P, P', Q' will touch the ellipse at Q and Q'.

Hence Q, Q' are the points at which PP' subtends the greatest and least angles respectively.

46. Draw the tangent Qr.

Then, since the angles PSQ, QSr are equal, Q always lies on the tangent at the end of the focal chord RSr.

47. The triangle YCY' will be the greatest possible when YCY' is a right angle: P will then lie on the circle of which SS' is diameter.

This intersects the ellipse in four points, provided SS' is greater than BB'.

48. The points where the lines joining the foci of the two ellipses meet the common auxiliary circle are points through which the common tangents pass.

49. The circle passing through the feet of the perpendiculars is the auxiliary circle of the ellipse.

50. Draw QN perpendicular to AB.

Then $\quad QN : NA :: BP : AP :: CA : AT,$

and $\quad QN : BN :: AT : AB.$

Therefore $\quad QN^2 : AN\,.\,NB :: CA : AB.$

Therefore the locus of Q is an ellipse of which AB is major axis.

51. PG, GN, NP are at right angles to CD, DR, RC respectively.

Therefore the triangles CDR, PGN are similar.

Therefore $\quad PG : CD :: PN : CR :: BC : AC.$

52. Let PS, QS meet the ellipse and circle again in p, q.

And let $P'Cp'$ be the diameter parallel to SP.

Then, since pq is an ordinate,

$$SQ : SP :: Sq : Sp :: Qq : pP :: AA' : P'p'.$$

Again, $\quad PS.Sp : AS.SA' :: CP'^2 : CA^2.$

Therefore $\quad SR.Pp : 2BC^2 :: P'p'^2 : AA'^2,$

or $\quad Pp : AA' :: P'p'^2 : AA'^2.$

Therefore $\quad SQ : SP :: AA' : Qq,$

and $\quad Qq = P'p'.$

53. If SP meet $S'Y'$ in L', $SL' = AA'$;

therefore $\quad SR = AC.$

54. Since the directions before and after impact are equally inclined to the tangent at the point of impact, the lines in which the ball moves will touch a confocal ellipse or hyperbola.

55. Let the tangent at P meet the tangents at A and A' in F and F'.

Then, since the angles PSF, FSA and PSF', $F'SA'$ are respectively equal, S (and similarly S') lies on the circle of which FF' is diameter.

56. P', D', the two angular points, will lie in PN, DM respectively.

Therefore $\quad P'N : NC :: DM : NC :: BC : AC.$

Therefore P' lies on a fixed straight line through C.

Similarly Q' lies on the other equi-conjugate diameter.

57. The angles SPS', STS' are equal by Ex. 15.

$\therefore \quad SPS' : STS' :: SP.S'P : ST.S'T :: CD^2 : ST^2.$

58. If T be the centre, then, since the angles TSP, TSA are equal, T lies on the tangent at A.

59. Let QL be the ordinate of Q.

Then $$QL : LS :: CN : NP,$$

and $$QL : LS' :: CM : MD.$$

$$QL^2 : SL . SL' :: CM . CN : PN . DM :: AC^2 : BC^2,$$

or Q lies on an ellipse of which SS' is minor axis.

60. If P is the corresponding point on the ellipse, and SZ the perpendicular on the tangent to the circle,

$$SZ : AC :: CT - CS : CT :: AC^2 - CS . CN : AC^2$$
$$:: SP : AC;$$
$$\therefore SZ = SP.$$

61. The tangent at Q is parallel to the normal at P; $\therefore$ the tangent at P is parallel to the normal at Q.

62. If $PQ\ P'Q'$ be the parallelogram, the angle DHE is the supplement of $HPQ' + HQP'$;

that is, of $SPQ + SQP$; that is, of DSE.

Hence S, H, D, E lie on a circle.

63. Let the line through C parallel to the tangent meet the directrices in Z, Z'.

Since the auxiliary circle is fixed, SY, $S'Y'$ are fixed straight lines meeting ZZ' in fixed points y y'.

And $$Cy . CZ = CX . CS = CA^2.$$

Therefore Z and Z' are fixed points.

64. The angle $S'TZ = STY = SZY =$ complement of YZT.

Therefore YZ and $S'T$ are at right angles.

65. If G be the centre of the circle, GL bisects SP at right angles.

Therefore SP is equal to the latus rectum.

66. If CZ be perpendicular to YY', the perimeter of the quadrilateral is equal to SS' together with twice $CZ+ZY$, which is greatest when $CZ=ZY$, that is when SPS' is a right angle.

67. Draw SZ perpendicular to $S'Z$ the straight line on which S' lies.
Let $PS'P'$ be the chord parallel to SZ.

Produce PP' both ways to M and M', so that

$$S'M=S'M'=AA'.$$

Then the lines drawn through M and M' perpendicular to SZ are fixed, and $SP=PM$, $SP'=P'M'$.
Hence the ellipse will touch two parabolas having S for focus.

68. Let TQ, TQ' be tangents, V the middle point of QQ'.
Then $QV.VQ'=CP^2-CV^2=CV.VT$.
Hence Q, Q', C, T lie on a circle.

69. Draw QM perpendicular to the minor axis.
Then $QC^2 : AC^2-CN^2 :: BC^2 : AC^2$,
or $QC^2 : BC^2 :: AC^2-CN^2 : AC^2$.
Therefore $BC^2-CN^2-QN^2 : CN^2 :: BC^2 : AC^2$,
or $BC^2-MC^2 : QM^2 :: AC^2+BC^2 : AC^2$.
Therefore Q lies on an ellipse of which BB' is minor axis.

70. If QM be the ordinate of Q,

$$AM^2 : CN^2 :: QM^2 : PN^2 :: AM.MA' . AC^2-CN^2.$$

Therefore $AM.AA' : AM^2 :: AC^2 : CN^2$,
or $2CN^2=AC.AM$.
But $AQ.AO : CP^2 :: AM.AC : CN^2$;
therefore $AQ.AO=2CP^2$.

71. $SP : SN :: SC : CQ :: SC : AC-QR.$

Therefore $SP . AC = SP . QR + SN . SC.$

But $SP : XS + SN :: SC : CA$

or $SP . AC = XS . SC + SN . SC.$

Therefore $SP . QR = XS . SC = BC^2.$

72. If the tangent meet the tangent at A in T, and $S'Y$, $S'Z$ be perpendicular to TS, and the tangent, T, A, Y, S', Z lie on the circle of which $S'T$ is diameter.

The angles YTZ, ATS' are equal since ATS, $S'TZ$ are equal. Art. 68.

Therefore the chords YZ, AS' on which these angles stand are equal.

73. If P, Q, P', Q' be the parallelogram, p, q, p', q' the points of contact, pq, $p'q'$ are parallel focal chords bisected by PCP'.

But QCQ' bisects pp', qq' and is therefore conjugate to PCP' and parallel to pq, $p'q'$.

Therefore $CQ = CQ' = CA$

74. If T be the point from which the tangents are drawn,

ST, $S'T$ are perpendicular to TP', TP respectively.

Therefore SP, $S'P'$ are both parallel to CT.

75. $CS^2 : CA^2 :: CG : CN :: CG . CT : CN . CT.$

Therefore $CG . CT = CS^2.$

76. If PG be the normal at the point of contact,

$$CG . CT = CS^2.$$

Therefore G is a fixed point and P lies on the circle of which GT is diameter.

77. Let the given straight line pq meet the axis t.

Let the tangents at p and q meet in Q.

Let CQ meet pq in V and the curve in P

Through P, Q draw PG, QG' perpendicular to pq, meeting the transverse axis in G and G'.

Then $$CG' : CG :: CQ : CP :: CP : CV :: CT : Ct;$$
$$\therefore CG' . Ct = CG . CT = CS^2,$$
or G' is a fixed point.

78. Draw SY perpendicular to the tangent; produce SY to L making $LY = YS$.

The point of intersection of the circles described with centres L and P', and radii AA' and $AA' - SP$ respectively will be the second focus.

79. Draw SY, SY' perpendicular to the given tangents.

The point of intersection of circles described with centres Y and Y', and radius equal to CA will be the centre.

80. If OS be drawn perpendicular to PQ, S will be one focus.

If SP, PS' be equally inclined to OP and SQ, QS' to OQ, S' will be the other.

Bisect SS' in C, and take CA in SS' such that
$$2CA = SP + PS'.$$
If X be the foot of the directrix,
$$CX . CS = CA^2.$$

81. $$Qq : Aq :: PN : AN,$$
and $$Rr : rA' :: PN : NA'.$$
Therefore $$Qq . Rr : Aq . A'r :: PN^2 : AN . NA'$$
$$:: BC^2 : AC^2 :: SL : AC.$$
Now $$Aq : qA' :: Aq^2 : Qq^2,$$
and $$A'r : rA :: A'r^2 : Rr^2.$$
Therefore $$Aq . A'r : Ar . A'q :: AC^2 : SL^2.$$

82. By Ex. 75. $$CT : CS :: CS : CG;$$
therefore $$TS : CS :: SG : CG.$$
But $$TY : PY :: TS : SG;$$
$$\therefore TY^2 : PY^2 :: CS^2 : CG^2 :: CT : CG :: TZ : PZ.$$

83. If O be the intersection of the lines

$$OC^2=AC^2+BC^2.$$

84. $$TP : TQ :: CP' : CQ',$$

and the angles PTQ, $P'CQ'$ are equal.
Therefore PQ is parallel to $P'Q'$.

85. S, P, t, S' lie on a circle, and the triangles SCt, PYS are similar.

Therefore $$St : Ct :: SP : SY :: CD : BC,$$

or $$St.PN : Ct.PN :: CD.BC : BC^2.$$

Therefore $$St.PN=CD.BC.$$

86. If the tangent at Q meet the minor axis in t', the angle $SQS'=St'S'$, or t' is on the circle.

Now $$QM.Ct'=BC^2=PN.Ct.$$

Therefore $$QM : PN :: Ct : Ct' :: Ct : Ct+St$$
$$:: BC : BC+CD$$ by Ex. 85.

87. If $S'Z$ be perpendicular to TY,
the angle STY = complement of TZY', (Ex. 64.)
= half supplement of YCY' the angle at the centre,
$=CYY'$; and $STY=S'TY'$.

88. The tangents at L and L' are perpendicular to the tangent at P, and therefore D and D' where they meet the tangent at P are on the director circle.

Now $$DL : DP :: D'L' : D'P;$$

therefore PQ bisects the angle LPL'.
Therefore $LP+PL'$ = diameter of director circle.

89. AB, AE are equally inclined to BC,

and $$AB^2=AD.AE.$$

Therefore AB is a tangent.

90. If the tangent at P meet the tangent at A in T, TS, TS' bisect the angles PSA, $PS'A$.

91. If the chords of intersection NO, PQ meet in T and CD, CE, $C'D'$, $C'E'$ are parallel radii,

$$CD^2 : CE^2 :: TN.TO : TP.TQ :: C'D'^2 : C'E'^2.$$

92. If PN meet CD in K,

$$PK : PQ :: SG : SP' :: SA : AX,$$

and $$PN.PK = PG.PF = BC^2.$$

Therefore PQ varies inversely as PN.

93. Draw perpendiculars SY, CE, $S'Y'$, SZ, CF, $S'Z'$ on tangents TP, TQ,

then $CT^2 = CF^2 + TF^2 = CZ^2 + TZ.TZ' = CA^2 + SY.S'Y'$
$= CA^2 + CB^2.$

94. If the circle meet the minor axis in K and L, the tangents at P and Q meet either in K or L, see Ex. 15.

95. This problem is equivalent to Ex. 45.

96. Let CV bisecting the chord QSQ' meet the curve in P and directrix in T.

Let DCD' be the parallel diameter.

Then $SR : SC :: CS - CR : CS :: CT - CV : CT$

$:: CS^2 - CG^2 : SC^2 :: SG.GS' : CS^2$

$:: SP.PS' : CA^2 :: CD^2 : CA^2 :: QQ'^2 : DD'^2$

by Art. 76.

97. Let the tangent at Q meet PN in P' and the axis in U.

Then $$CT.CN = CA^2 = CM.CU;$$

therefore $$CT : CM :: CU : CN,$$

or $$TM : CM :: NU : CN.$$

But $$PN.Q'M : Q'M^2 :: TN : TM,$$

or $$PN.Q'M : CM.MU :: CN.NT : CM.NU.$$

Hence $PN.QM : CN.NT :: CM.QM : CM.P'N$

$:: QM^2 : P'N.QM.$

Now $PN^2 : CN.NT :: BC^2 : AC^2 :: QM^2 : Q'M^2.$

Hence $$PN.Q'M : PN^2 :: Q'M^2 : P'N.QM.$$

Therefore $P'N : PN :: AC : BC$,

or P' is on the auxiliary circle.

98. The diameter bisecting PQ is fixed, hence V the centre of the circle, is a fixed point.

VM bisecting RS at right angles, is a fixed straight line;

PQ and RS are equally inclined to the axis;

$\therefore$ CM and CV are equally inclined to the axis.

Therefore M is a fixed point, and RS a fixed straight line.

99. The angle BSC

$$=BAC+SBA+SCA=BAC+HBC+HCB$$
$$=BAC+\text{supplement of } BHC.$$

Hence if BHC is constant, BSC will be constant.

100. The angles SPT, HPt are each equal to SQH;

also $$STP=tQS-PtH$$
$$=HPt-PtH=PHt.$$

Therefore $$TP : SP :: HP : Pt,$$

or $$TP\,.\,Pt=SP\,.\,HP=CD^2.$$

Therefore CT, Ct are conjugate.

101. CT bisects PQ and is parallel to SP;

$\therefore$ T is the foot of the perpendicular from S' on PT. See Cor. (3), Art. (66).

102. $SC : CY$ is a given ratio and SY is fixed.

103. Take p a point near and let the focal chord $p'Sq'$ meet pq in O;

$$PQ : p'q' :: pO\,.\,Oq : p'O\,.\,Oq' :: ST\,.\,Oq : Sp'\,.\,Oq'$$
$$:: ST\,.\,pq : Sp\,.\,p'q' \text{ ultimately.}$$

104. Dropping perpendiculars from the focus on the sides, their feet are the middle points, and, as they lie on a circle, form a rectangle; the diagonals, intersecting in H, are therefore at right angles, and SAD can be proved equal to HAB.

105. If CD, CP meet the directrix in E and G, ES is perpendicular to the chord of contact of tangents from E, which is parallel to CP.

106. If CP, DC meet the tangent at A in Q and R, prove that $AQ \cdot AR = BC^2 = AS \cdot AS'$.

Then $QSA = ARS'$, and $QRA = AQS'$.

CHAPTER IV.

THE HYPERBOLA.

1. If the circle whose centre is P touch the circles whose centres are S and H in Q and R,

$$SP \sim HP = SQ \sim HR.$$

Therefore P lies on an hyperbola of which S and H are foci.

2. $SD^2 = BC^2 = CS^2 - CA^2 = CS^2 - CX \cdot CS = CS \cdot SX.$

Therefore the triangles SCD, SDX are similar.

3. If the straight line meet the curve in P and the directrix in F,

$$SF : SX :: CS : CA :: SA : AX :: SR : SX.$$

Therefore $\qquad SF = SR.$

Draw PK perpendicular to the directrix.

Then $\qquad PF : PK :: SC : CA :: SP : PK.$

Therefore $\qquad SP = PF.$

4. Draw SD, SD' perpendicular to the asymptotes.

Then DD' is the directrix.

5. If the asymptote meet the directrix in D, then DS drawn at right angles to CD meets the axis in the focus.

6. If PK, QL be the perpendiculars from the given points on the directrix $PS-SQ=PK-QL$ which is constant.

Therefore S lies on an hyperbola of which P and Q are foci.

7. If the circle inscribed in the triangle ABC touch the sides in D, E, F; B, C, D being given,

$$BA-CA=BF-EC=BD-DC.$$

Hence A lies on an hyperbola of which B and C are foci and D a vertex.

8. $PN : Pg :: SY : SP :: BC : CD$

$$:: PF : AC :: AC : Pg.$$

Therefore $$PN=AC.$$

9. Draw CD parallel to the given line and CP parallel to the tangent at D.

Then the tangent at P is parallel to CD and the given line.

10. Let $A'P$ and $P'A$ meet in Q, and draw the ordinate QM.

Then $$QM : A'M :: PN : NA',$$

and $$QM : AM :: P'N : NA.$$

Therefore $$QM^2 : AM.MA' :: PN^2 : AN.NA'$$

$$:: BC^2 : AC^2,$$

or Q lies on an hyperbola having the same axes.

11. Let the tangent at P, AP and $A'P$ meet the minor axis in t, E and E'.

Then $CE : PN :: CA : AN :: CA.A'N : AN.NA'$,

and $$CE' : PN :: CA.AN :: AN.NA'.$$

Hence $CE-CE' : PN :: 2CA^2 :: AN.NA'$.

Now $$PN^2 : AN.NA' :: PN.Ct : AC^2;$$

therefore $$CE-CE'=2Ct.$$

Therefore Pt bisects every line perpendicular to AA' terminated by $A'P$, AP.

12. SPT is an isosceles triangle since pST is parallel to SP.

Therefore $$SP = ST.$$

13. Draw SD perpendicular to the asymptote and SK parallel to it.

If TS bisect the angle PSK, T being on the asymptote, TP is the tangent at P. Draw SY perpendicular to it.

Then CM which bisects DY at right angles will meet the asymptote in the centre C. DX drawn perpendicular to CS will be directrix.

14. If the tangent at P meet the tangents at A and A' in Z and Z', ZS and $Z'S$ are the internal and external bisectors of the angle ASP.

Hence the foci lie on a circle of which ZZ' is diameter.

15. Draw PK perpendicular to the directrix and DS at right angles to the asymptote.

Draw cxs at right angles to the directrix meeting it in x.

With centre P and radius PS such that $SP : PK :: cs : cD$, describe a circle meeting Ds in S.

Then S is the focus.

16. Draw Qq', Pp and Rr, Qq parallel to the asymptotes.

Then $CP : CQ :: Cp : Cq' :: Cq : Cr :: CQ : CR$.

17. $$QC^2 - CB^2 : CN^2 - CA^2 :: BC^2 : AC^2$$
$$:: PN^2 : CN^2 - CA^2.$$

Therefore $$PN^2 = QB \,.\, QB'.$$

18. $DN : NA :: QM : MA$ and $EN : NA'$
$$:: QM : MA'.$$

Therefore $ND \,.\, NE : AN \,.\, NA' :: QM^2 : AM \,.\, MA'$
$$:: PN^2 : AN \,.\, NA',$$

or $$PN^2 = ND \,.\, NE.$$

19. A, A', Y, Z lie on the auxiliary circle.

Therefore $$AT \,.\, TA' = YT \,.\, TZ.$$

20. If the tangent at P meet the asymptotes in L and L', $CD=PL=PL'$; therefore Q divides LL' in a constant ratio.

Draw $QH.QK$ parallel to the asymptotes.

Then $QH.QK$ varies as $CL.CL'$ and is therefore constant.

Therefore Q lies on an hyperbola having the same asymptotes.

21. This is equivalent to the preceding.

22. Since $SK=S'K$, K lies on the circle passing through S, S' and P, and since KPt is a right angle, t lies on the same circle.

Therefore $GK : S'K :: SG : SP :: SA : AX$,

and $St : tK :: SY : SP :: BC : CD$.

23. Let P, P' be the points of trisection of the arc SS' and let XM bisect SS' at right angles, then $SP=2PM$ and $S'P'=2P'M$.

Hence P and P' lie on hyperbolas of which S and S' are foci and XM directrix.

If C, C' be the centres $CS=4CX$ and $C'S'=4C'X$.

Therefore C and C' are the points of trisection of the chord SS'.

24. Draw SZ parallel to the asymptote: the angle $STQ=TSZ=TSP$.

Therefore $SQ=QT$.

25. Since the hyperbolas have the same asymptotes the ratios $CS : BC : CA$ are constant.

Let NP be the fixed line parallel to an asymptote, and PQ proportional to an axis.

Then PQ^2 varies as CS^2, that is, as $CN.NP$, that is, as NP.

Hence Q lies on a parabola having NP for a diameter.

26. $PY.PY' = AC^2 - CP^2 = CD^2 - BC^2 = CS^2$
$$- \left(\frac{SP - S'P}{2}\right)^2 = CS^2 - CA'^2 = CB'^2.$$

27. Let TP meet the other asymptote in T', then $PT = PT'$.

Therefore $$PQ = R'P = QR.$$

28 Draw OrR parallel to PQ, meeting the ellipse and hyperbola in r and R.

Let Oa, Ob be the axes, then since OP, Or are conjugate in the ellipse, and OQ, OR in the hyperbola, if PN, QM, rl, RL be the ordinates,

$$ON.Ob = rl.Oa;\ PN.Oa = Ol.Ob;\ QM.Oa = OL.Ob;\ OM.Ob = RL.Oa.$$

Therefore $$PN : ON :: QM : OM$$

since $$rl : Ol :: RL : OL,$$

or OP and OQ are equally inclined to the axes.

29. Through S draw SC parallel to the bisector of the angle between the asymptotes meeting the asymptote which is given in position in C.

Draw SD perpendicular to that asymptote, and DX to CS. Then if A be taken in CS such that $CA^2 = CX.CS$, A is vertex.

30. If the tangents at P and Q meet in T, then since the perpendiculars from T on SP, SQ, HP, HQ are all equal, a circle can be described with centre T to touch SP, SQ, HP and HQ.

31. If CL, CL' be the asymptotes, S will lie in the bisector of the angle LCL'.

Draw PL, PL' parallel to the asymptotes to meet them in L and L';

and take S in CS such that $CS^2 = 4CL.CL'$,

then S is a focus.

32. If the conjugate diameters PCP', DCD' be given, complete the parallelogram $LML'M'$ formed by the tangents at D, P, D' and P'.

The diagonals LL', MM' are the asymptotes and the axes bisect the angles LCM, LCM'.

33. Let QT and RQ meet the asymptotes in L and M.

Then $\quad QL : PH :: RH : TL :: CR : CT$

$$:: RM : TK :: PK : QM;$$

therefore $\quad QL . QM = PH . PK,$

or Q is on the curve.

34. Let CL bisecting the angle ACB' meet PN in L, draw QM parallel to LC.

Then CL is proportional to CN;

therefore $CL . CM$ is proportional to $CN . NT + CN^2$, that is to CA^2.

Hence Q lies on an hyperbola of which CL and CB are asymptotes.

35. Draw SY perpendicular to the tangent and produce it to Z making $YZ = SY$.

Then if Q be the point of contact, and P the fixed point,

$$HP - PS = HQ - QS = HZ;$$

therefore $\quad HP - HZ = PS,$

or the locus of H is an hyperbola of which P and Z are foci.

36. If PT, Pt the tangents to the ellipse and hyperbola meet BC in T and t, then since the curves have the same conjugate axis, for $\quad CA^2 = CS^2 + CB^2$

$$Ct . PN = BC^2 = CT . PN,$$

or $\quad CT = Ct.$

37. This problem is the converse of Ex. 3.

38. If G be the point of intersection

$$CG = \tfrac{2}{3}\, CP,$$

or G lies on an hyperbola having the same asymptotes.

39. The angle $CYY' = S'PY' = S'NY'$ since S', P, N and Y' lie on a circle.

Therefore Y, Y', C and N lie on a circle.

40. If SY meet $S'P$ in Z, $SY = YZ$; therefore $S'Y$ bisects PG.

Similarly SY' bisects PG.

41. $BC^2 : AC^2 :: NG : CN :: CT.NG : CN.CT$;

therefore $$CT.NG = BC^2.$$

42. The angle $STP = TS'P + S'PT = TS'P + SPT$

$$= PS'S + SS't = \text{supplement of } PSt.$$

43. $$Pt.PT = CD^2 = SP.S'P,$$

or $$SP : PT :: Pt : S'P;$$

and the angles SPT and tPS' being equal, the triangles SPT, tPS' are similar.

44. The circles SCE, $S'CE'$ stand upon equal chords SC, $S'C$ and contain equal angles SEC, $S'E'C$, since CE is parallel to the bisector of SPS'.

45. If the tangent at P meet the tangent at A, the vertex of the branch on which P lies, in T, T is the centre of the circle inscribed in the triangle SPS', since TS, TS' bisect the angles ASP, $AS'P$.

46. $CT : CA :: CA : CN :: CP : CQ$, or AQ is parallel to PT.

47. $$CE : CA :: CS : CA;$$

therefore $CE = CS$; but $CD = CA$.

Therefore AD and SE are parallel.

48. If E be the centre of the circle and EK its radius,

$$EK : CE :: BC : SC,$$

or $$EK : CA :: BC : SC+BC$$
$$:: (SC-BC)\,BC : CA^2.$$

And $$SR' : CS :: BC : AC :: SR : BC;$$

therefore $$RR' : SR :: CS-BC : BC,$$

or $$RR' : CS-BC :: SR : BC :: BC : CA.$$

Therefore $$EK=RR'.$$

49. $$PM=PL;$$

therefore $$GL=GM.$$

50. If Ca, Cb be the conjugate diameters and one hyperbola touch the ellipse in P, the tangent at P will meet Ca, Cb in T, t, such that $TP=Pt=CD$.

Hence PD is bisected by Ct,

and tD touches the other hyperbola and is parallel to CP.

51. If LL' and MM' be the tangents,
$$CL : CM :: CM' : CL',$$
or LM' and $L'M$ are parallel.

52. If the tangent at P meet the tangents at A and A' in F and F' and QM be the ordinate of the centre of the circle,
$$QM : MS :: SA : AF,$$
and $$QM : MS' :: S'A' : A'F'.$$

Hence $QM^2 : SM.MS' :: SA^2 : AF.A'F' :: SA^2 : BC^2$

(Art. 126).

Hence the locus of Q is an hyperbola of which S and S' are vertices.

53. If PM be perpendicular to the directrix,

$$PK : PM :: CS : CA :: SA : AX :: SP : PM,$$

or $$PK=SP.$$

54. Let PD meet an asymptote in n, draw Pl, Dm parallel to Cn.

Then $$Dm \,.\, Dn = Pn \,.\, Pl;$$
therefore $$Dn = Pn.$$
Therefore if LPL' is tangent at P, LD is tangent at D, and CP, CD are conjugate.

55. If QR meet the asymptotes in q and r, $qQ = rR$; therefore if EPe be the tangent at P,
$$CL : CN :: qQ : Pe :: PE : qR :: CN : CM.$$

56. If the circle intersect the axis in b, B,
$$CB \,.\, Cb = CS^2$$
or $$CB^2 + CB \,.\, Bb = CA^2 + CB^2;$$
therefore $$CB \,.\, Bb = CA^2.$$

57. Let the straight line $q'Q'APQq$ meet the asymptotes in Q', Q.
Draw RCR' parallel to AP terminated by $A'q$, $A'q'$.
Then $$PQ' = AQ = CR = Qq,$$
and $$Q'q' = CR' = AQ' = PQ;$$
therefore $$Pq' = Pq.$$

58. T is the centre of the circle inscribed in the triangle $PS'Q$, therefore the difference between PTQ and half $PS'Q$ is a right angle.

59. Draw CD, CE parallel to OA, OB and PH, PK parallel to and terminated by CE, CD.
Then $PH : OD :: CK : CE :: CP : CA :: CB : CP$
$$:: CD : CH :: OE : PK;$$
therefore $$PH \,.\, PK = OD \,.\, OE,$$
or P lies on an hyperbola having CD, CE for asymptotes.

60. Draw PH, PK, QH', QK' parallel to the asymptotes.
Then $PL : QM :: PH : QH' :: QK' : PK :: QN : PR,$
or $$PL \,.\, PR = QM \,.\, QN.$$

61. If TK, TN be perpendicular to the directrix and SP, $TK = SN$.

Therefore $ST : TK :: ST : SN$ a constant ratio,

and the angle between the asymptotes is double PST, that is, double the external angle between the tangents.

62. $$Q'V^2 - RV^2 = CD^2 = RV^2 - QV^2,$$

or $$QV^2 + Q'V^2 = 2RV^2.$$

Again $$CT \cdot CV = CP^2 = CV \cdot CT';$$

hence $$CT = CT'$$

63. If V be the middle point of PQ, then since R, V are the middle points of LRL' and $LPQl$, RV is parallel to the asymptote $CL'l$.

Hence $$PM + QN = 2RE.$$

64. If TP, TQ be the tangents, PTQ, STS' have the same bisector which passes through the point where the circle meets BCB'.

65. The tangents at P and Q intersect in t on the circle and BCB'.

Hence the angle $PtQ = PSQ$.

66. The angle $PNY = PSY = PS'Y' =$ supplement of PNY'.

67. The triangle YCY' is greatest when YCY' or SPS' is a right angle.

In that case PT meets BC in t such that $Ct = CS$;

therefore $$CS \cdot PN = Ct \cdot PN = BC^2.$$

68. The triangle SPS' : triangle StS' :: $SP \cdot PS' : St^2$ :: $CD^2 : St^2$.

69. $S'P$ is parallel to CY and $S'Q$ to CZ.

Therefore $S'T$ is parallel to bisector of YCZ and is perpendicular to YZ.

70. If G be the centre of the circle, GL bisects SP;

therefore $$SP = 2PL = 2SR.$$

71. The tangent at Q is parallel to the normal at P, therefore the tangent at P is parallel to the normal at Q, or CP is conjugate to normal at Q.

72. If Y be the point from which the tangents are drawn, SP and $S'P'$ are both parallel to CY.

73. $SC^2 : AC^2 :: CG : CN :: CG . CT : CN . CT$,

or $$CG . CT = SC^2.$$

74. By Ex. 73, G the foot of the normal is a fixed point;

therefore P lies on the circle of which TG is diameter.

75. If TP, TQ be the tangents, CT will bisect PQ in V,

and $$CT . CV = CT^2,$$

or PQ is a tangent at V.

76. Let GQ meet the conjugate in G'.

Then $QG' : QG :: CN : NG :: AC^2 : BC^2$.

Therefore, by Art. 111, QG' is normal at Q.

77. If PM be drawn perpendicular to the directrix of the parabola the angle $PTQ = SPT - SQT =$ half $SPM -$ half $SQS' =$ half $SS'Q$.

78. If $abcd$ be the quadrilateral and S lie on the circle the angle $Hcd = Scb = Sab = Had$,

or H is on the circle.

79. If PP' be the chord of contact and CV bisect PP' then CV, PP' are parallel to a pair of conjugate diameters in both conics.

Hence if from a common point Q, a double ordinate QVQ' be drawn parallel to PP', Q' must lie on both curves.

Similarly RR' the line joining the other two common points is parallel to PP'.

80. If SD, SD' are perpendiculars from the common focus on the asymptotes, DD' is the tangent at the vertex of P and a directrix of H.

If P be a common point, and PM perpendicular to DD',

$$SP : PM :: SC : CA,$$

but $$SP = PM + SX.$$

Therefore $SP : SX :: CS : CS - CA :: CS : AS.$

Hence $$AS . SP = SX . CS = BC^2 = AS . SA',$$

or $$SP = A'S.$$

Therefore $A'P$ touches the parabola at P.

81. With centre P, the given point and radius of the given length describe a circle meeting the other asymptote in p.

Then $pPQq$ is the line required.

82. Let CB, CA be semiaxes of the ellipse, Ca, Cb of the hyperbola.

Let PN meet the asymptote in Q,

then $$QN^2 : CN^2 :: Cb^2 : Ca^2,$$

or $$QN^2 + CN^2 : Ca^2 + Cb^2 :: CN^2 : Ca^2;$$

but $$SP + S'P = 2CA,$$

and $$SP - S'P = 2Ca.$$

Hence $4CA . Ca = SP^2 - S'P^2 = SN^2 - S'N^2 = 4CN . CS;$

therefore $$CA^2 : CS^2 :: CN^2 : Ca^2 :: QN^2 + CN^2 : Ca^2 + Cb^2 :: CQ^2 : CS^2.$$

Therefore Q lies on the auxiliary circle of the ellipse.

83. Let Q be a common point.

Then $SQ - QH = AA'$ and $SQ - QP = SP - 2PH$
$$= AA' - PH.$$

Therefore $$QP = QH + PH,$$

or Q must be the other extremity of the focal chord PH.

84. If $A'K$ meet the directrix in F, then,
since $$SA'=2A'X,$$
$FA'S$ is an isosceles triangle and FS is parallel to KD.

Also $A'F : FP :: A'X : XN :: A'S : SP$
or FS bisects the angle $A'SP$;

therefore if SP and DK meet in Q, QSD is an isosceles triangle.

Therefore Q lies on the circle of which $A'D$ is diameter.

85. This problem is a particular case of Ex. 61.

86. $$PL \,.\, PL'=PL^2=CD^2=PG \,.\, Pg;$$
therefore G, g, L, L' lie on a circle of which Gg is diameter.

C is on this circle since GCg is a right angle.

The radius of this circle varies as Gg and therefore as CD and therefore inversely as the perpendicular from C on LL'.

87. If PCP', DCD' be conjugate diameters and Q any point on the curve,
$$QP^2+QP'^2=2CP^2+2CQ^2;\ QD^2+QD'^2=2CD^2+2CQ^2.$$
Therefore
$$QP^2+QP'^2-QD^2-QD'^2=2CP^2-2CD^2=2AC^2-2BC^2.$$

88. If $S'L'$, $S'M'$ be drawn parallel to the asymptotes LS', MS' bisect the angles $PS'L'$, $PS'M'$.

Hence $LS'M=$ half the angle between the asymptotes.

89. If PT meets the tangent at A in V, VS bisects the angle ASP;

therefore $SP : ST :: PV : VT :: AN : AT.$

90. If P is a point of intersection, let the tangent and normal of the ellipse at P meet the transverse axis in T and G, and the conjugate axis in t and g.

Then, PT being the normal of the hyperbola, the semi-axes of which are $A'C$ and $B'C$,

$$CT : CN :: SC^2 : A'C^2, \qquad \text{Art. 111},$$

$$\therefore AC^2 : CN^2 :: SC : A'C;$$

$$\therefore CN.SC = AC.A'C.$$

Again, Pg being the normal of the ellipse,

$$Cg : PN :: SC^2 : BC^2, \qquad \text{Art. 72},$$

and

$$Cg.PN = B'C^2,$$

$$\therefore B'C^2 : PN^2 :: SC^2 : BC^2$$

and

$$PN.SC = BC.B'C.$$

Hence, if PN meet the asymptote in Q,

$$QN : CN :: B'C : A'C,$$

and it is easily deduced that

$$QN : PN :: AC : BC.$$

91. Let $ABCD$ be the quadrilateral, A, B, and C being fixed points.

Then $$AB + CD = BC + AD,$$

or $$CD - DA = CB - AB.$$

Hence D lies on an hyperbola of which A and C are foci.

92. Since Q, S, C, t lie on a circle, the angle

$$tQC = tSS' = SPt,$$

hence CQ is parallel to SY and CY, SQ are equally inclined to SY;

therefore $$SQ = CY = CA.$$

93. Draw SY perpendicular to the tangent and produce to Z making $SY = YZ$.

Then if P be the point of contact $HZ = HP - SP = AA'$.

Hence the locus of H is a circle of which Z is centre.

94. RS and VS' bisect the angles PSQ and $PS'Q$; let QS, $S'P$ meet in Z.

Then $RSP + VS'Q =$ half $PSQ +$ half $PS'Q =$ half QSP $+$ half $SZS' -$ half $SQS' = QSP +$ half $SPS' -$ half SQS'

$$= QTP + TQS - SQT = PTQ.$$

95. CgP is an isosceles triangle, and the angle

$$CGt = CPT = TCP;$$

therefore $PG = Ct$ and $CD^2 = PG \,.\, Pg = Cg \,.\, Ct = CS^2$.

96. Since the asymptote CD bisects BA, CD is parallel to the axis of the parabola and BA is parallel to the other asymptote.

If $QPVP'Q'$ parallel to BA meet CD in V,

$$QV = VQ' \text{ and } PV = VP';$$

therefore $$QP = Q'P'.$$

97. Let EL be the ordinate of E, and draw EF perpendicular to PN.

Then, CD being conjugate to CP, the triangles CDM and PFE are similar and equal.

$$\therefore CL = CN + EF = CN + DM,$$

$$\therefore CN : CL :: AC : AC + BC,$$

and similarly $$EL : PN :: BC : BC - AC;$$

$$\therefore EL^2 : (BC - AC)^2 :: PN^2 : BC^2$$
$$:: CN^2 - AC^2 : AC^2$$
$$:: CL^2 - (AC + BC)^2 : (AC + BC)^2.$$

98. If PM be drawn from the centre perpendicular to BC

$AP : PM :: PC : PM$, a constant ratio;

therefore P lies on an hyperbola of which A is focus and BC directrix.

If S be the other focus and SP meet the circle in Q

$$SQ = SP - PA = \text{constant},$$

or, the envelope is a circle of which S is centre.

99. The conics will be confocal having their foci H and H' on PG,

such that $$PH^2 = PT \,.\, Pt = CD^2.$$

For their locus see Ex. 9 on the ellipse.

100. If $SY, SZ, S'Y', S'Z'$ be perpendiculars on tangents at right angles

$$CT^2 - CA^2 = TY \cdot TY' = SZ \cdot SZ' = CB'^2.$$

If SYZ, $S'Y'Z'$ are perpendicular to parallel tangents and CWW' be the perpendicular through the centre

$$2CW = SY + S'Y'; \; 2CW' = SZ - S'Z',$$

and $$SY - SZ = S'Y' + S'Z';$$

$$\therefore 4CW'^2 + 4CB'^2 = 4CW^2 - 4CB^2.$$

Hence $$CW^2 - CW'^2 = CB^2 + CB'^2.$$

101. The chord QR is inclined to the axis at the same angle as the tangent at P and is therefore always parallel to a fixed line.

102. TP and the asymptote subtend equal angles at S';

$$\therefore PS'T = S'TC = STP.$$

103. $$SF^2 = FX^2 + SX^2 = CF^2 - CX^2 + SX^2$$
$$= CF^2 + CS^2 - 2CS \cdot CX$$
$$= CF^2 + CS^2 - 2CA^2 = CF^2 - CA^2 + CB^2$$
$=$ square of tangent from F
$$= FA \cdot FB.$$

104. If S be the focus of the ellipse and S' of the hyperbola,

$$CS : CA :: CA : CS';$$

$\therefore$ S and S' coincide with the feet of the directrices.

The relation, $CN \cdot CT = CA^2$, proves that S and S' are the feet of the ordinates, and the relation, $Ct \cdot PN = BC^2$, proves that t and t' are on the auxiliary circle.

Also $$Ct' : CS = AC : CS = CS' : Ct;$$

$\therefore$ the tangent intersects at right angles.

105. For $PG \cdot Pg = CD^2$, Art. 123, $= PL^2 - PL \cdot PL'$, and the diameter of the circle is Gg, which varies as CD, Art. 123, and therefore inversely as CY.

CHAPTER V.

THE RECTANGULAR HYPERBOLA.

1. THE angle $PCL = CLP =$ complement of LCY.

2. $QV^2 = VP \,.\, Vp$, hence VQ touches at Q the circle QPp.

3. $$LP = PM = CD = PG = Pg.$$
Hence LGM is a right angle.

4. If LM be the straight line, and C be the corner of the square, $CL \,.\, CM$ is constant, hence LM touches an hyperbola of which CL, CM are asymptotes.

5. Let AP, $A'P'$ meet in Q, and draw the ordinate QM.

Then $$QM : MA :: PN : NA,$$
and $$QM : MA' :: P'N : NA'.$$
Hence $$QM^2 : AM \,.\, MA' :: PN^2 : AN \,.\, NA';$$
therefore $$QM^2 = AM \,.\, MA'.$$
Hence Q lies on a rectangular hyperbola having AA' for transverse axis.

6. If P, P' be joined to Q meeting an asymptote in R and R',
the angle
$$QRL = CLP - QPL = LCP - PP'Q = CR'P' = QR'L.$$

7. Produce LP to M making $PM=PL$, then if MC be drawn perpendicular to the given asymptote CL, C is the centre. In CS the bisector of the angle LCM take S such that CS is a mean proportional between CL and CM.

Then S is focus, and X the middle point of CS is the foot of the directrix.

8. The angle $DCL=PCL$,

and $D'CL=P'CL$;

hence $DCD'=PCP'$.

9. A diameter is a mean proportional between the parallel focal chord and AA', therefore focal chords parallel to conjugate diameters are equal.

10. As in the preceding, focal chords at right angles are equal, since diameters at right angles are equal.

11. If CD, Cd be conjugate to CP in the ellipse and hyperbola,

$$CD^2=SP.PS'=Cd^2=CP^2.$$

12. $$PN=CM;\ DM=CN;$$

and $$CD=CP.$$

13. $$SP.PS'=CD^2=CP^2.$$

14. $QV^2=CV^2-CP^2=CV^2-CT.CV=CV.VT.$

Hence QV touches the circle CTQ.

15. If D be the intersection of tangents at A and B,

$$CD^2=AC^2+BC^2=SC^2.$$

Hence D lies on the circle of which SS' is diameter.

16. If LPM, $G'PG$ be the tangent and normal to one,

$$LP=PM=CD=CP=PG=PG';$$

therefore GG' is the tangent to the second hyperbola.

17. The angle $CRT=CQT+RTQ=2CLQ+LTL'$

$$=CLM+CL'T=CLR'+L'CQ'=TR'Q'.$$

Hence C, T, R and R' lie on a circle.

18. $QR^2 = CN^2 = CA^2 + RN^2 = CA^2 + CQ^2 = AQ^2$.

19. Let AQ, AR be the fixed straight lines, and P the middle point of QOR.

Through C the middle point of AO draw CH, CK parallel to AQ, AR,

and through P draw PHM, PKN parallel to AR, AQ.

Then the complements AC, HK about the diagonal MN are equal.

Therefore $PH \cdot PK$ is constant, and P lies on a rectangular hyperbola, having CH and CK for asymptotes.

20. Draw QB perpendicular to AB, and make AB equal to CD, then A is a fixed point.

Then $\quad AD : AB :: BC : CD :: QB : DP,$

or $$PD \cdot DA = AB \cdot BQ.$$

Hence P lies on a rectangular hyperbola of which AB is one asymptote.

21. If D, E, F, O be the centres of the escribed and inscribed circles,

$$OC \cdot CF = DC \cdot CE,$$

since the triangles OCE, DCF are similar.

Hence the hyperbola is rectangular since diameters at right angles are equal.

22. If the diameters of the parallelogram $LML'M'$ meet in C, the angle $SLS' = SL'S'$.

Hence S and S' lie on a rectangular hyperbola circumscribing $LML'M'$. (Art. 137.)

23. If PSq, SQq be the chords and D be a point on the directrix, such that DS bisects the angle QSp, then D will lie in pq.

But DS is perpendicular to the asymptote since Pp, Qq are equally inclined to it, therefore D lies on the asymptote.

Similarly Pq, Qp meet in D', the foot of the perpendicular from S on the other asymptote.

24. If V be the middle point of PQ and POP' be a diameter,

the angle $\quad VNP = NPV = QP'P = POV.$

Hence O, P, V, N lie on a circle.

If OQ meet the given tangent in T, produce OQ to V, making OV a third proportional to OT and OQ; with centre V and radius, a mean proportional between VO and VT, describe a circle meeting the given tangent in P its point of contact. In the tangent measure off

$$PL = PM = OP,$$

then OL and OM are the asymptotes.

25. Draw PD perpendicular to the base QR,

then, since $\quad PD^2 = DQ \,.\, DR,$

DP is the tangent at P.

26. Let a circle on DE meet the hyperbola in P and Q, draw the diameters PCP', QCQ'.

Then, since the angle DPE is either equal or supplementary to $DP'E$ and DQE to $DQ'E$, the similar circle on the other side of DE, will meet the curve in P' and Q'.

27. Let OAD, OBC be the fixed straight lines, PM, PN, PL perpendiculars from the centre of the circle on BC, AD and the bisector of the angle AOB.

Let PM, PN meet OL in m, n.

Draw Ll, Ll', Pr, Pr' perpendicular to BC, AD, Ll, Ll' respectively.

Then, $\quad BM^2 + MP^2 = AN^2 + NP^2,$

or $\quad PN^2 - PM^2 = BM^2 - AN^2$

which is constant,

$PN^2 = (Ll' + Lr')^2 = (Ll - Lr)^2 + 4Ll' \,.\, Lr' = PM^2 + 4Ll \,.\, Lr.$

Hence the rectangle $Ll \,.\, Lr$ is constant;

but $Ll : OL$ in a constant ratio and $Lr : PL$ is constant.

Therefore $PL \,.\, LO$ is constant, or P lies on a rectangular hyperbola having OL for an asymptote.

28. If P be the point of contact

$$CL=2PN,\ CL'=2CN;$$

therefore $CL\,.\,CL'=2Ca^2=4PN\,.\,CN.$

Hence $CN\,.\,CL' : Ca^2 :: Ca^2 : PN\,.\,CL,$

or $AC^2 : Ca^2 :: Ca^2 : CB^2.$

29. Draw CV conjugate to PQ.

Then, the angle

$$TPQ=PP'Q=PCV=CPQ';$$

therefore the angles CPQ, TPQ' are equal.

30. Let DB, DC meet the asymptotes in b and c: draw AH, AK parallel to the asymptotes. Then OBb, OAK are similar triangles, also OCc, OAK.

Hence

$$OB : OA :: Ob : OK :: OH : Oc :: AK : Oc :: OA : OC;$$

therefore A lies on the circle of which BC is diameter as does D.

31. Let the tangents at P and Q meet the asymptote in L and M.

The angle

$$PCQ=PCL-QCM=PLC-CMT=LTM$$
$$=\text{supplement of } PTQ.$$

32. Each hyperbola passes through the orthocentre of the triangle ABC.

Hence D is that orthocentre.

Now the line joining the middle point of AB to the middle point of CD is a diameter of the nine point circle.

And $AB^2+CD^2=$square on diameter of circumscribed circle.

Hence the circles intersect at right angles.

33. $PN^2=CN^2-CA^2=CN^2-CN\,.\,CT=CN\,.\,NT.$

Hence the triangle CPN is similar to PTN, and therefore to tTC.

34. This problem is a particular case of Ex. 61 on the hyperbola.

35. CM, CN which bisect PP', PQ' are conjugate being equally inclined to the asymptotes;

therefore $P'Q'$ is a diameter.

36. If AB be a diameter of the hyperbola, CD subtends, at A and B, angles which are both equal and supplementary, and are therefore right angles.

37. If AQ', BQ meet in R, the angle QAQ' = supplement of $QBQ' = RBP$;

therefore R lies on the circle.

38. Let CV, CV' bisect PQ, $P'Q$.

The angle $PRQ = PQL = VCQ = CQV'$,

so $P'R'Q = CQV$.

Draw VM perpendicular to CQ,

then $PQ : QR :: VM : CV$,

and $P'Q : R'Q :: VM : QV$.

Now $P'Q = 2CV$,

and $PQ = 2QV$;

therefore $QR = QR'$.

39. Let PN meet CF in K,

then CF varies as CG, and PF varies as PK, or Ct, or CT.

Hence $PF.FC$ is proportional to CG, CT or CS^2.

Hence P lies on a rectangular hyperbola having CF for asymptote.

40. If the tangent at Q meet in V the line joining the fixed points A and B,

$$VQ^2 = VA . VB.$$

41. If the chord QR meet the tangent at P in E,

$$RPL = QPE = PRQ.$$

42. If D be the point,

$$CD \,.\, CT = 2AC^2 \text{ and } CD = AC.$$

43. $CP = CD$ and are equally inclined to the asymptote.

44. Let BAD be the given difference, and draw CL parallel to AD, meeting BA in L. CAL, CBL are similar triangles;

$$CL^2 = AL \,.\, BL.$$

45. If tQT, $t'QT'$ be the tangents, and SM perpendicular to the axis,

$$\begin{aligned} SQM &= SQT - MQT = S'Qt' - CtT \\ &= QS'M + QT'C - CtT = QS'M; \\ \therefore\ QM^2 &= SM \,.\, S'M. \end{aligned}$$

46. If V is the middle point of OP,

$$\begin{aligned} OTV &= VOT, \because OTP \text{ is a right angle,} \\ &= OCV, \text{ Art. 136;} \end{aligned}$$

$$\therefore\ O, V, C, T \text{ are concylic, and}$$

$$\therefore\ VCT = VOT = OCV.$$

Similarly, if U is the middle point of OQ,

$$OTU = UTQ.$$

CHAPTER VI.

THE CYLINDER AND THE CONE.

1. TAKE two points E and A on the generating line, and draw EX at right angles to the axis, making EX equal to EA.

Then the plane containing AX, and perpendicular to the plane EAX will cut the cylinder in an ellipse of the required eccentricity.

2. Take two points E and A on the generating line and with centre A, and radius twice EA, describe a circle meeting EF in X.

Then the plane through AX perpendicular to the plane EXA will intersect the cone in an ellipse of the required eccentricity.

3. Take two points EA on the generating line, the least angle of the cone will be, when EA is double the perpendicular from A on EF, that is when the semi-vertical angle is equal to the angle of an equilateral triangle.

4. A tangent plane to a cone touches it along a generating line OF, hence OF is parallel to all sections parallel to the tangent plane which are therefore parabolas.

If C be the centre of the sphere FES, the ratios $CS : CA$ and $CA : CO$ are constant, and the angle OCS is constant, therefore COS is constant, and S lies on a cone of which O is vertex and OC axis.

5. Through the flames of the candles which are treated as points, draw planes intersecting in the ceiling, in a straight line, since these fixed planes must always be tangent planes to the ball, the locus of the centre of the ball is a horizontal straight line.

6. The triangles AEX, $A'E'X'$ have all their sides parallel;

therefore

$$SA : AX :: EA : AX :: E'A' : A'X' :: S'A' : A'X'.$$

7. If C be the centre of the sphere FES, the angle OCS is constant.

And the ratios $CE : CA$, $CE : CV$, and $CS : CA$ are all constant;

therefore $CS : CV$ is constant and the angle CVS is constant, or VS is a fixed straight line.

8. Take two points E and A on the generating line and with centre A and radius AX, such that $EA : AX$ in the ratio of the eccentricity, describe a circle intersecting FE in X, the section of which AX is axis will have the required eccentricity.

9. XS, XS' are tangents to the same sphere FES of which C is centre.

Let SS', EF meet the axis in V', L, and let CX meet SS' in M.

Then $$CL . CV' = CM . CX = CE^2.$$

Hence V and V' coincide.

10. Draw CN perpendicular to the axis,

then $$2CN = A'D' - AD,$$

and $$2ON = OD + OD'.$$

In CN take a point Q, such that

$$QN : CN :: DO^2 : DA^2;$$

then since $OD' - OD : 2CN :: OD : DA,$

$2QN : OD' - OD :: DO : DA;$

also $AD + A'D' : 2ON :: AD : DO,$

and $AA'^2 = DD'^2 + (AD + A'D')^2.$

Hence $QO^2 : CA^2 :: OD^2 : DA^2.$

Hence Q lies on a sphere of which O is centre, and therefore C lies on a spheroid, having OD for its axis, which is oblate or prolate according as DO is greater or less than DA, that is according as the vertical angle of the cone is greater or less than a right angle.

11. Draw a plane CT through C, the centre of the sphere perpendicular to the axis intersecting the tangent plane at S in T. Then since the angles COE and CTS are equal, and $CE = CS$, it follows that $CT = CO$.

Hence S lies on the surface generated by the revolution of the circle of which CT is diameter.

12. Only two circles can be described passing through S, and touching the generating lines in which the plane through S and the axis intersects the cone.

If ST, ST' be the tangents, and OS meet the circles in D and D',

the angle $DST = \text{half } SCD = \text{half } SC'D' = D'ST'.$

Hence the planes of the corresponding sections make equal angles with OS.

13. If the plane of section meet the plane through O perpendicular to the axis in the line ZZ', and PK be perpendicular to ZZ'

OP or OQ bears to PK a constant ratio.

Hence if P' in the projection corresponds to P

$OP : P'K$ in a constant ratio.

But OP' is equal to the perpendicular from P on the axis which bears to OP a constant ratio.

Hence the ratio $OP' : P'K$ is constant.

14. $A'Q - QA = SS'$ and a right cone can be constructed of which Q is vertex, such that the generating lines intersect the ellipse.

Hence

$$PQ + AS = AS + QR + RP = AE + EQ + SP = AQ + SP.$$

15. Through the vertical straight line which is the locus of the luminous point draw vertical tangent planes to the ball intersecting the inclined plane in OY, OZ. The locus of C the centre of the shadow will be the straight line bisecting YZ, at right angles, SY, SZ being perpendiculars from the point of contact of the ball on OY, OZ which are tangents to the elliptic shadow.

16. The given plane to which the sections are perpendicular must be supposed to contain the axis of the cone.

Take OK on the generating line equal to AS.

Draw KG perpendicular to OK, meeting a line through O at right angles to the axis in G, then G is a fixed point.

Draw GM perpendicular to AS,

then $\quad CE : EO :: OK : KG,$

or $\quad KG : KA :: AE : CE.$

Hence the angle $KAG = ACE =$ half KAM.

Hence AS touches a circle centre G and radius GK.

17. $VP = VQ = VA + AQ = 2AE + AN = 2AS + AN.$

18. EX', $X'A'$ will be parallel to EX, XA.

Hence if AF be drawn parallel to $A'E'$, the ratio of the eccentricities is $AE : AF$ which is constant.

19. The volume varies as the area AVA' and BC; and the area AVA' varies as $AV \cdot VA'$, or $AD \cdot A'D'$, that is, BC^2.

Hence BC is constant.

20. $A'O - OA = A'E' - EA = A'S - SA = SS'$.

Hence the locus of O is an hyperbola of which A and A' are foci.

21. $BC^2 = EC \cdot CE'$.

Hence the locus of C is an hyperboloid of revolution generated by the revolution round the axis of the cone of an hyperbola of which OE, OE' are asymptotes.

22. CA and EA are the bisectors of the angle OAS.

Hence the sphere on CE as diameter, intersects the plane of section in a circle of which AA' is diameter.

CHAPTER VII.

CURVATURE.

1. SL being a fourth of the chord of curvature at L through S, LG the normal is a fourth of the diameter of curvature.

2. The normal and tangent at L are equally inclined to the axis.

3. Draw SO at right angles to SP, meeting the normal at P in O, and from O draw OQ at right angles to AP; then the chord of curvature through A is $4PQ$; drawing AZ perpendicular to TP,

$$PQ : OP :: AZ : AP,$$

or $$PQ.AP = OP.AZ.$$

Again, $$SP : PO :: AZ : AT,$$

or $$OP.AZ = SP.AT;$$

also $$PY : SP :: AT : TY,$$

or $$PY^2 = SP.AT = PQ.AP.$$

Hence $$4PQ : 4PY :: PY : AP.$$

4. The diameter at P and SP are equally inclined to the normal at P; hence chord of curvature parallel to the axis is equal to $4SP$.

5. Let QM be the ordinate of Q.

Then $$QM : MG :: PN : NG,$$
or $$PN . MG = 2AS . QM = pn . Mg.$$
Also $$Nn = Gg.$$
Hence $$MG : Mg :: pn : PN,$$
or $$MG : Gg :: pn : pn - PN.$$
Therefore $$MG : pn :: Nn : pn - PN :: pn^2 - PN^2 : 4AS\,(pn - PN) :: pn + PN : 4AS.$$
But $$QM : MG :: PN : 2AS;$$
hence $$QM : 2PN :: pn\,(pn + PN) : 16AS^2.$$

If P and p approach each other indefinitely, Q is the centre of curvature,

and $$PQ : PN + QM :: PG : PN.$$
But $$QM : PN :: PN^2 : 4AS^2 :: AN : AS;$$
hence $$QM + PN : PN :: AN + AS : AS :: SP : AS.$$
Hence $$PQ : PG :: SP : AS :: PG^2 : SR^2.$$

6. If PK be perpendicular to the directrix and S' the farther focus,

$$SP : PK :: CA : CX;$$
hence $$SP = \tfrac{1}{2}AC,$$
and $$S'P = \tfrac{3}{2}AC.$$
Therefore $$PE . PS' = \tfrac{3}{2}AC^2 = 2SP . S'P = 2CD^2,$$
or the circle of curvature passes through S'.

7. If PV be the chord of curvature through the focus and pp' the focal chord parallel to the tangent at P,

$$PV . AC = 2CD^2,$$
or $$PV . AA' = DD'^2 = AA' . pp';$$
hence $$PV = pp'.$$

8. Let Q be the middle point of the chord and QM its ordinate.

If PNP' be the double ordinate, $P'Q$ is a diameter.

If PQ meet the axis in W,

$$WM = TN = NW;$$

hence $$AM = 5AN,$$

and $$QM^2 = PN^2 = 4AS \,.\, AN = \tfrac{4}{5}AS \,.\, AM;$$

or Q lies on a parabola of which A is vertex and AS axis.

Produce SA to S' making $S'A = 3AS$, and let PQ meet the tangent at A in Y'.

Then $AY'^2 = \tfrac{9}{4}PN^2 = 9AS \,.\, AN = 3S'A \,.\, AN = S'A \,.\, AW$.

Hence $S'Y'$ is perpendicular to PQ, and PQ envelopes a parabola of which S' is focus and A vertex.

9. DR and PCP' are equally inclined to the axis, and D, R, P, P' lie on a circle;

hence PR, DP' are equally inclined to the axis.

So DQ, PD' are equally inclined to the axis;

hence PR, DQ are parallel, since DP', PD' are parallel.

10. $$PG = CD = CP.$$

Hence the radius of curvature varies as CP^3.

11. $$LP \,.\, PC = PT \,.\, Pt = CD^2.$$

Hence PL is equal to half the chord of curvature in direction PC.

$$CP \,.\, CL = CP \,.\, PL + CP^2 = CD^2 + CP^2 = AC^2 + BC^2.$$

12. The circle on PE as diameter touches the curve at P and goes through Q; when Q coincides with P the circle becomes the circle of curvature.

13. If the tangents at two near points P and Q meet in T,

$$TP : TQ :: CD : CE.$$

Hence the difference of TP and TQ is very small compared with either, and if a circle be described, of which the intersection of normals at P and Q is centre, to touch TP and TQ at P and Q, when P and Q coincide this becomes the circle of curvature at P.

14. If C be the centre of curvature at the vertex,

$$AC = 2AS.$$

If PR be the tangent,

$$PR^2 = CP^2 - CR^2 = PN^2 + CN^2 - 4AS^2$$
$$= 2AC \, . \, AN + CN^2 - AC^2 = AN^2.$$

15. PCP' and the tangent at P are equally inclined to the axis.

16. If O be the centre of curvature,

$$PO^2 = AC \, . \, C = PF \, . \, CD.$$

But $$PO \, . \, PF = CD^2,$$

hence $$PO = CD = PF.$$

Hence, if with centre C and radius CP, such that

$$CP^2 = AC^2 + BC^2 - AC \, . \, BC,$$

a circle be described, it will meet the curve in points, at which the radius of curvature has the required value.

17. If PV be the chord and CQ be drawn parallel to Pt,

$$2CD^2 = PQ \, . \, PV = Ct \, . \, PV.$$

But $$Ct \, . \, PM = BC^2,$$

hence $$PV : PM :: 2CD^2 : BC^2.$$

18. If PQ be the common chord of the ellipse and circle of curvature, TPt must make equal angles with both axes, or $CT = Ct$.

Make the angle ACP such that

$$PN : NC :: BC^2 : AC^2;$$

then CP will meet the ellipse in the point required.

19. SP is one-fourth of the chord of curvature through S, hence PQ is half the radius of curvature.

20. If PV be the chord,

$$PV.PD=2CD^2.$$

But $$PN=ND,$$

hence $$PV : CD :: CD : PN.$$

21. $$PG : PF :: PK : PC :: PE : Pg,$$

or $$PF.PE=PG.Pg=CD^2;$$

hence E is the centre of curvature.

22. Draw SQ at right angles to SP to meet the normal in Q; let the tangent at P meet the directrix in Z.

Then $$PL : PS :: ZP : ZS :: PQ : PS.$$

Therefore $PL=PQ=$half the radius of curvature.

23. If CE bisect PQ, PCE is a right angle and

$$PF.PE=CP^2=CD^2;$$

hence PQ is equal to the diameter of curvature at P.

24. $$OP : CP :: PQ : PF,$$

or $$OP.PF=CP^2=CD^2;$$

hence O is the centre of curvature at P.

25. If the normal at P meet BC in K; S, P, H, K lie on a circle; hence, when P coincides with B, K becomes the centre of curvature at B.

26. If HT be produced to H' making $H'T=TH$, TQ, PH' will be parallel;

hence $$PR : RS :: H'T : TS :: TH : TS.$$

Therefore $PR : PS :: TH : 2TC :: HP : 2CY :: HP : 2CA$,

or $$2PR.AC=PS.HP=CD^2.$$

Hence PR is one-fourth of the chord of curvature through S.

27. If O be the centre of curvature at A,

$$SO : SA :: SA : AX.$$

Hence curvature of ellipse is greater than that of parabola, and curvature of parabola is greater than that of hyperbola.

28. By Ex. 6, $\qquad SP = \frac{3}{2}AC,$

and $\qquad SP : CY' :: PE : PC.$

Hence $\qquad PE : EP' :: 3 : 1.$

29. Let CQ' be conjugate to CF and CP' parallel to PQ.

Then $\quad OE^2 : OF^2 :: OQ.OP : OF^2 :: CP'^2 : CQ'^2$

$$:: CD^2 : CQ^2 :: TP^2 : TF^2 :: TE^2 : TF^2.$$

Hence $TEOF$ is cut harmonically.

30. Project the angle between the common diameters into a right angle; the ellipses obtained will be inscribed, symmetrically, in a square, and will therefore be equal.

31.
$$\begin{aligned} SE^2 &= Py^2 + (EP - Sy)^2 \\ &= SP^2 + EP^2 - 2EP.Sy \\ &= EP^2 - 3.SP^2. \qquad \text{Art. 160.} \end{aligned}$$

32. Let F be the middle point of PE, the radius of curvature at P.

Then $\qquad PF.SY = SP^2, \qquad$ Art. 160.

$$\therefore SY : SP :: SP : PF,$$

$\therefore$ SYP, SPF are similar triangles, and the angle PSF is a right angle, so that the locus of S is the circle, diameter PF.

CHAPTER VIII.

PROJECTIONS.

1. THE theorems are obtained by projection from the following properties of the circle.

Art. (65) Every diameter of a circle is bisected at the centre, and the tangents at its extremities are parallel.

(70) If the chord of contact of tangents from T to a circle meet CT in N, $CT \,.\, CN = CA^2$.

(71) If M, t correspond to M', t' on the circle,

$$CM : CM' :: BC : AC :: Ct : Ct';$$

therefore $$CM \,.\, Ct = BC^2.$$

(73) If the chord of contact of tangents from T to a circle meet CT in V and CT meet the curve in P,

$$CT \,.\, CV = CP^2.$$

(74) A diameter of a circle bisects all chords parallel to the tangents at its extremities.

(75) If a diameter of a circle bisects chords parallel to a second, the second diameter bisects all chords parallel to the first.

(76) Art. 77 is meant.

If PCP', DCD' be diameters of a circle at right angles,

$$QV^2 : PV \,.\, VP' :: CD^2 : CP^2.$$

(78) If CA, CB be radii at right angles,

$$CN^2 = AM . MA', \; CM^2 = AN . NA',$$
$$CM : PN :: AC : BC :: CN : DM.$$

(81) The area of a square circumscribing a circle is constant and equal to the rectangle contained by diameters at right angles.

(82) If CD, CP be radii at right angles and the tangent at P meet a pair of radii at right angles in T and t,

$$PT . Pt = CD^2.$$

(83) Cor. 1. The two tangents TP, TQ from any point are equal, and the parallel diameters ACA', BCB' are equal,

$$\therefore TP : ACA' :: TQ : BCB;$$

and these ratios are unaltered by projection.

Cor. 2. In the circle TCT' is a right angle,

and $$\therefore PT . PT' = CP^2 = CD^2,$$

CD being parallel to TPT'.

(84) Taking ABA' as a semicircle, ECF is a right angle; project on any plane parallel to the line ACA'.

2. If a parallelogram be inscribed in a circle its sides are at right angles. The greatest rectangle than can be inscribed in a circle is a square having its area equal to $2AC^2$; hence the greatest parallelogram that can be inscribed in an ellipse has its area equal to $2AC . BC$.

3. The theorem is true in the case of a circle, and follows by projection.

4. The greatest triangle which can be inscribed in a circle is an equilateral triangle of which C is the centre of gravity.

Produce PC to V, making $2CV = PC$;

then, if QVQ' be the ordinate, PQQ' is the greatest triangle which can be inscribed in the ellipse having its vertex at P.

5. If a straight line meet two concentric circles, the portions intercepted between the curves are equal.

6. The locus of the point of intersection of tangents at the extremities of diameters of a circle at right angles is a concentric circle.

7. The locus of the middle points of lines joining the extremities of diameters of a circle at right angles is a concentric circle.

8. If CP, CD be radii of a circle at right angles and CA bisect the angle PCD, the tangent at A meets CP in T such that $PD^2=2AT^2$.

9. If a chord AQ of a circle be produced to meet the diameter at right angles to CA in O and CP be parallel to AQ,

$$AQ \cdot AO=2CP^2.$$

10. If OQ, OQ' are tangents to a circle and R be a diagonal of the parallelogram of which OQ, OQ' are adjacent sides, then if R be on the circle the locus of O is a concentric circle.

11. If a parallelogram be inscribed in a circle and from any point on the circle straight lines are drawn parallel to the sides of the parallelogram, the rectangles under the segments of these lines made by the sides are equal to one another.

12. If a square circumscribe a circle and a second square be formed by joining the points where its diagonals meet the circle, the area of the inner square is half that of the outer. And if four circles be inscribed in the spaces between the outer square and the circle, their centres will lie on a concentric circle.

13. If a rectangle be inscribed in a circle so that the diameter bisecting one pair of sides is divided in a constant ratio, the area is constant.

14. If a parallelogram circumscribe a circle and one of its diagonals bear a constant ratio to the diameter it contains, the area is constant.

15. PQR is a triangle inscribed in a circle, the centre being the intersection of lines joining the angular points to the middle points of opposite sides. If PC, QC, RC meet the circle again in P', Q', R', the tangents at P', Q', R', will form a triangle similar to PQR, its area being four times as great.

16. The locus of the middle points of chords of a circle passing through a fixed point is a circle of which the line joining that point to the centre is diameter.

17. The ellipse which touches the middle points of the sides of a square (i.e. a circle) is greater than any other inscribed ellipse.

18. If a polygon circumscribe a circle, its area is a minimum when any side is parallel to the line joining the points of contact of adjacent sides.

19. The greatest triangle which can be inscribed in a circle has one side bisected by a diameter and the others cut in points of trisection by the diameter at right angles.

20. AB is a given chord of a circle, C any point of the circle, the locus of the intersection of the straight lines joining A, B, C to the middle points of BC, CA, AB is a circle.

21. If CP, CD are radii of a circle at right angles, the circle on PD as diameter will go through C.

22. The theorem is true in the case of a circle intersecting a concentric rectangular hyperbola, and follows generally by projection.

23. If V is the middle point of Qq, project CVQ into a right angle.

24. If PT, pt are tangents at the extremities of a diameter Pp of a circle, then if any diameter meet PT in T and the diameter at right angles meet pt in t, and any tangent meet PT in T' and pt in t',

$$PT : PT' :: pt' : pt.$$

25. If CP, CD be radii of a circle at right angles and Pp, Dd be drawn parallel to any tangent, and any line through C meet Pp, Dd and the tangent in p, d and t,

$$Cp^2 + Cd^2 = Ct^2.$$

26. If ACA', BC and CD, CP be pairs of radii of a circle at right angles and if BP, BD be joined, also AD, $A'P$, the latter intersecting in O, $BDOP$ is a parallelogram.

27. If TM be perpendicular to SP,

$$TM : TP :: SY : SP :: BC : CD;$$

hence $\qquad TP : CD$ is constant.

If a point be taken on a tangent to a circle such that its distance from the point of contact is constant and therefore proportional to the parallel radius, its locus is a concentric circle.

CHAPTER IX.

CONICS IN GENERAL.

1. $TN : XN :: SR : SX :: SP : XN$;

hence $$SP = TN.$$

Also $TP \,.\, TP' = TN^2 - PN^2 = SP^2 - PN^2 = SN^2$.

2. Draw Pm, Qn perpendicular to the directrix.

Then $PR : QN :: KP : KQ :: Pm : Qn :: SP : SQ :: PM : QN$,

or $$PR = PM.$$

3. $PS \,.\, SQ : AS \,.\, SA' :: Cp^2 : CA^2$,

and $$PQ \,.\, SR = 2SP \,.\, SQ.$$

Hence PQ varies as Cp^2.

4. Let Pp, Qq intersect in O.

Then $QO \,.\, Oq : PO \,.\, Op :: TP^2 : TQ^2 :: QO^2 : PO^2$.

Hence TO bisects pq as well as PQ, and is a diameter and goes through t.

5. Since RS is the exterior bisector of the angle $P'SQ'$,

$$SP' : SQ' :: RP' : RQ'.$$

6. Let $S'T$, ST meet PQ in EE', and let TF, TG be the common interior and exterior bisectors of the angles ETE', PTQ.

Bisect FP in O.

Then $$OE.OE' = OF^2 = OP.OQ.$$

Now $$RP : RQ :: PE : EQ,$$

and $$R'P : R'Q :: PE' : E'Q.$$

Again, $$OP : OE :: OE' : OQ;$$

hence $$PE : OE :: E'Q : OQ.$$

Also $$OP : OE' :: OE : OQ;$$

hence $$PE' : OE' :: EQ : OQ.$$

Therefore $$RP.PR' : RQ.QR' :: PE.PE' : EQ.QE'$$
$$:: OE.OE' : OQ^2 :: OF^2 : OQ^2.$$

Hence TF bisects the angle RTR', and the angles RTP, $R'TQ$ are equal.

7. TS and KS are the interior and exterior bisectors of the angle PSP'.

Hence if ST meet PP' in E,

$$RK : TE :: KP : EP :: KP' : EP' :: KR' : ET,$$

or $$RK = KR'.$$

8. If DE, $D'E'$ are perpendicular to SP, SP',

then $$SE = SE'.$$

Hence DE, $D'E'$ intersect on the bisector of the angle PSP', which is ST.

9. If PP' meet the directrix in K, PP' is harmonically divided at S and K.

Hence any chord through S is harmonically divided by the directrix and the tangents at P and P'.

10. Draw SY perpendicular to the tangent, then since
$$SC : CY :: SA : AX,$$
the locus of C is a circle.

11. Let Pp meet the curve in P', and let QP' meet the directrix in q'.

Then since pS, $q'S$ are the exterior bisectors of the angles PSP', QSP', the angle $pSq' =$ half $PSQ = pSq$; hence q and q' coincide.

12. $SL : SP :: FT : FP :: TN : PK$.
Therefore $SL : TN :: SP : PK :: SA : AX$.

13. $PN^2 : AC^2 - CN^2 :: CB^2 : CA^2 :: Cb^2 : Ca^2$

$$:: pn^2 : Cn^2 - Ca^2.$$

Hence, if $PN = pn$, $AC^2 - CN^2 = Cn^2 - Ca^2$,

or
$$CN^2 + Cn^2 = CA^2 + Ca^2.$$

14. $QR : LG :: PQ : PG :: PM : PN$.
Hence $QR : PM :: LG : PN :: SG : SP :: SA : AX$.

15. Draw KV parallel to the axis.

Then $VK : VP :: GS : SP :: SG' : SQ :: VK : VQ$,

or
$$PV = VQ.$$

Also $PL : PM :: PK : PG :: PV : PS$;

hence $2PL . PS = PM . PQ = SR . PQ = 2SP . SQ$,

or
$$PL = SQ.$$

Hence $SV = VL$, and the diagonal of the parallelogram SL goes through V.

16. If PQR be the triangle and S the focus, make the angles QSr, QSp each equal to the supplement of PSR.

Then, if $PSq = PSr$, p, q and r are the points of contact.

17. By Ex. 20, Chap. I., if KV be drawn parallel to the axis, $PV = VQ$.

Hence $PN : PL :: PK : PG :: PV : PS$.

Hence $2PN . PS = SR . PQ = 2SP . SQ$,

or
$$PN = SQ.$$

Hence $SN = 2SV$, and the locus of N is a similar conic.

18. Let CT, CT' meet the curve in p, d.

Then $$CT.PR=2Cp^2,$$
and $$CT'.QR=2Cd^2.$$
Hence the triangle
$$CTT' : 2 \text{ triangle } Cpd :: 2Cpd : PRQ,$$
or the triangle
$$CTT' : AC.BC :: AC.BC : \text{the triangle } PRQ.$$

19. Let S be the centre of the circumscribed circle, H the ortho-centre,

then the feet of the perpendiculars from S and H on AB, BC, CA lie on the nine-point circle,

and the angle SAB = complement of $C = HAC$.

Therefore with S and H as foci a conic can be inscribed in ABC.

20. Let SV meet the directrix in Q and PK in Z, let QP meet the axis in C.

Then $$PZ : SP :: SG : SP :: SA : AX;$$
hence $$PZ : PK :: SA^2 : AX^2.$$
Now $$SC : CX :: PZ : PK :: SA^2 : AX^2,$$
or C is the centre.

21. $DE.DF : DC^2 :: AB^2 : AC^2 :: DG^2 : DC^2$,

or $$DG^2 = DE.DF.$$

22. By Ex. 74 on the parabola, if PGQ be the chord of contact,
$$DF : FG :: PG : GQ :: GF : FE,$$
or $$FG^2 = FD.FE.$$

23. If Epq be drawn parallel to DTP,
$$DP^2 : Ep.Eq :: DF : EF.$$

For DF is parallel to a generating line VM of the cone of which the hyperbola is a section.

Draw Dlm, LEM in the plane $VFEM$ to meet the cone.

Then the sections of the cone by the parallel planes, lPm, pLQ are similar,

and $$Dm = EM.$$

Hence

$$DP^2 : Ep.Eq :: Dl.Dm : EL.EM :: Dl : EL :: DF : EF.$$

Again, $Ep.Eq : EQ^2 :: PT^2 : TQ^2 :: EK^2 : EQ^2$;

hence $EK^2 = Ep.Eq$ if Epq meet PQ in K.

And $DG^2 : GE^2 :: DP^2 : EK^2 :: DP^2 : Ep.Eq :: DF : EF.$

Therefore $$FG^2 = FD.FE.$$

24. Let the tangents at P, Q and R meet EB in p, q and r.

Then $Ep.Eq = EF^2$, if PQ meet EB in F;

also $EB^2 = Er.Ep$, $EC^2 = Er.Eq$.

Hence $EB^2 : EC^2 :: Ep : Eq$, a constant ratio.

By Ex. 23. the same proposition is true in the case of an hyperbola if EB be parallel to an asymptote.

25. GK being perpendicular to SP,

$$Pk : PK :: Pg : PG :: AC^2 : BC^2;$$

$$\therefore Pk \text{ is constant.}$$

Also $kL : Pk :: SG : SP$; $\therefore kL$ is constant.

26. Let the fixed line meet the curve in P and Q, and let the tangent at P meet SL in D, and the directrix in F; then, Art. 11, $SD : SF :: SL : SX$.

The angle FSP is a right angle, so that SF is a fixed line, and, SX being a fixed line, the ratio of SF to SX is constant; $\therefore SD$ is constant and D is fixed.

The envelope of PG is therefore the parabola, of which D is the focus and PQ the tangent at the vertex.

CHAPTER X.

HARMONICS, POLES AND POLARS.

1. If AOB be the common chord and $PQOpq$ any transversal,

$$PO \,.\, Op = AO \,.\, OB = QO \,.\, Oq.$$

2. OA is perpendicular to $B'C'$ and meets it in D,

$$OA \,.\, OD = \text{square on radius of concentric circle} = OB \,.\, OE = OC \,.\, OF.$$

Therefore $$OD = OE = OF.$$

3. Draw ABC to be bisected by OB in B, BEG to AO produced to be bisected by OC in E, and BFK to CO produced to be bisected by OA in F.

Then any one of the straight lines drawn through O parallel to AC, BG, BK will form a harmonic pencil with OA, OB, OC.

Draw BL, BM parallel to OA, OC to meet OC, OA in L and M.

Then since $OE = EL$, $OF = FM$, EF is parallel to LM and is therefore bisected by OB and is also parallel to ABC.

Hence the pencil BC, BE, BO, BF is harmonic.

4. Let the circles meet in P, bisect AC in E.

Then $$EB \,.\, ED = EC^2 = EP^2,$$

hence the circles cut at right angles.

5. By Art 182, PQ, AE, BD intersect in A, and $A\{B, E, Q, F\}$ is harmonic.

6. AP, BQ and PB, AQ meet each pair on the polar of O on which C lies;

And the pencil formed by CB, CO, CA and the polar of O is harmonic.

7. If A, B be points of contact and the third conic meet AB in C and D, A and B are the foci of the involution, P, Q are conjugate points.

Hence $ACBD$ is a harmonic range.

8. Let the common chords meet in E, and let $EPRQ$ be a tangent at R; then since the common chords are one conic of the system, E and R are foci of the involution $EPRQ$ and $EPRQ$ is a harmonic range.

9. Let TP, TQ be the tangents, TE any line, then F the pole of TE lies on PQ and $PEQF$ is a harmonic range.

10. If the tangent at P meet the asymptotes in L and L', $PL=PL'$.

Hence the pencil CD, CL, CP, CL' is harmonic.

11. A diameter is bisected at the centre: and the polars of the extremities of a diameter intersect at infinity.

12. If T be the pole of QR and H the second focus of the conic which touches the ellipse at Q,

$$PQ+QH=SQ+QS';$$

or
$$HS'=SP.$$

Therefore
$$HS'+S'P=S'P+SP;$$

or S' is on the conic of which H is focus.

Again the angles $TS'R$, $TS'Q$ are equal, and PS', $S'H$ are equally inclined to TS'; hence TS' is a tangent at S.

Therefore T lies on the directrix of the conic of which P, H are foci.

13. Let ST, $S'T$ meet PQ in E, E'; then the angles ETP, $E'TQ$ are equal.

Now the ranges $RPEQ$, $R'QE'P$ are harmonic and QTP is common to the two pencils; hence the angles $R'TQ$, RTP are equal.

14. The middle points of all chords of the cone parallel to the given line, lie in a plane through the vertex, let this plane meet the given line in P and any section through it in A and A'.

Then Q the pole of the given line lies in $AA'P$ and $AQA'P$ is a harmonic range. Since VA, VA' are fixed generating lines, VQ is a fixed straight line.

15. If Tpq be the chord, P its pole, then PN the ordinate of P is the polar of T.

Let CP meet pq in v and the curve in Q.

Let QM, QG' be the ordinate and normal at Q.

If PG be drawn perpendicular to pq, it is parallel to QG';

hence $\quad CG : CG' :: CP : CQ :: CN : CM$;

Therefore $\quad CG : CN :: CG' : CM :: SC^2 : AC^2$.

Hence G is a fixed point.

16. Let the polar of Q meet the conjugate CFD in R. Draw QQ' parallel to CP.

Then $\quad PE : PQ :: RF : PF$;

and $\quad PG : PQ :: CF : PF$;

hence $\quad EG : PQ :: CR : PF$;

or $\quad EG . PF = PQ . CR$.

Now Q is on the polar of R, since R is on the polar of Q.

Hence $\quad PQ . CR = CQ' . CR = CD^2$.

Hence EG is equal to the radius of curvature at P.

17. If O be the orthocentre of ABC and $A'B'C'$ the reciprocal triangle, $B'C'$, $C'A'$, $A'B'$ are perpendicular to OA, OB, OC respectively, and BC, CA, AB are perpendicular to OA', OB', OC' respectively.

Hence ABC and $A'B'C'$ have their sides parallel and O is the orthocentre of each.

18. If $pPSQq$ be the focal chord and the tangents at P and Q meet in T, TS is perpendicular to PQ, hence the tangents at p and q meet in T.

19. This theorem is the reciprocal of the following: if two circles intersect they have two common tangents: if one circle lie entirely within the other, they have no common tangents. Reciprocate with respect to a point on one circle and within the other.

20. If I, C be the centres of the inscribed and circumscribed circles, and CI meet them in r, r' and B, B' respectively, then if AA' be the major axis of the ellipse into which the circumscribed circle is reciprocated,

$$IA\,.\,IB = Ir^2,\quad IA'\,.\,IB' = Ir^2,$$
$$CI^2 = CB^2 - 2CB\,.\,Ir.$$

Hence $\quad IA : Ir :: Ir : CB - CI :: CB + CI : 2CB.$

and $\quad IA' : Ir :: Ir : CB + CI :: CB - CI : 2CB.$

Hence $\quad AA' : Ir :: 2CB : 2CB;$

or $\quad AA' = Ir.$

21. The four circles which circumscribe the triangles of a complete quadrilateral meet in a point.

22. See Ex. 30 on the parabola, or by reciprocation.

23. See Ex. 46 on the ellipse, or by reciprocation.

24. Let CPP', COO' be perpendicular to the polars of P and O.

Draw OX, OA perpendicular to CP and the polar of P;

PY, PB perpendicular to CO and the polar of O.

Then $\quad CO' : CP' :: CP : CO :: CY : CX.$

Hence $CO' - CY : CP' - CX :: CY : CX :: CP : CO$

or $\quad PB : OA :: CP : CO.$

25. See Ex. 18 on Chapter I.

26. (1) If a quadrilateral circumscribe a conic a pair of opposite sides subtend at the focus angles which are together equal to two right angles.

(2) If we reciprocate with respect to the focus S the new theorem is, if Q be taken on a circle and QL be drawn such that the angle SQL is constant, QL envelopes a conic of which S is focus.

27. The envelope of chords of a circle which subtend a constant angle at a fixed point on the circle is a smaller concentric circle.

28. Two circles, such that a point can lie within both cannot have more than two common tangents.

But if the circles be such that all points lie without both, or within one and without the other they may have four common tangents.

29. If a straight line meet the sides of the triangle $A'B'C'$ in L, M, N the circles circumscribing the triangles $A'B'C'$, $A'NM$, $B'NL$, $C'LM$ meet in a point.

30. If points P', Q' be taken on a circle of which C is the centre, $P'C$ will meet the line drawn through Q' at right angles to $P'Q'$ and $Q'C$ will meet the line drawn through P' at right angles to $P'Q'$ on the circle.

31. If S be the orthocentre of the triangle ABC and circles be described with centres A and B passing through C, S will lie on the radical axis of the two circles.

If we reciprocate with respect to S we see that if with the orthocentre of a triangle as focus we describe two conics each touching a side of the triangle and having the other two sides as directrices, the conics will have a parallel pair of common tangents and therefore their minor axes equal.

32. If a system of circles have two points in common the locus of their centres is a fixed straight line, and the polar of a fixed point meets the radical axis in a fixed point.

33. If the tangent and normal at P meet QR in T and G, the range $TRGQ$ is harmonic, since TP, PG bisect the angle QPR.

Hence PG is the polar of T.

Hence the pole of QR lies on PG since the pole of PG lies on QR.

34. If the tangents at P and Q meet in T and TA meet PQ in L, the range $DPLQ$ is harmonic ; hence the pencil TD, TP, TL, TQ and the range $DBAC$ are harmonic.

Therefore $ABDC$ is a harmonic range.

35. If the pencil joining $BPAQ$ to any point on the curve is harmonic, the pencil formed by joining them to any other point on the conic is harmonic.

For if BK, PK, AK, QK meet the directrix in $bpaq$, $bpaq$ is a harmonic range, provided $KEBPAQ$ be a harmonic pencil.

And the angles bSp, pSa, aSq, qSb are half the angles BSP, PSA, ASQ, QSB ;

Hence the pencil Sb, Sp, Sa, Sq is the same wherever K be taken on the curve.

Now PQ goes through O the pole of AB : let PQ meet AB in R.

Then if T be the pole of PQ, $TARB$ is a harmonic range.

Therefore the pencil joining Q to $BPAQ$ is harmonic ;

hence the pencil joining q to $BPAQ$ is harmonic.

Hence Pq bisects AB since AB, qQ are parallel.

36. Four circles can be described so as to touch the sides of a triangle, and the reciprocal of the radius of the inscribed circle is equal to the sum of the reciprocals of the radii of the other three.

If the triangle be equilateral the inscribed circle touches the three escribed circles.

37. If the tangents at P and Q meet the axes in T and V; the angle

$$PSQ = SQV - SPT = SVQ - STP = VST.$$

If SW be perpendicular to $P'Q'$, the tangents at the vertices intersect in W.

Draw SYZ perpendicular to the tangents at P and Q.

Then WSP', WSQ' are supplementary to WYP', WZQ'.

Hence $P'SQ'$, PSQ are supplementary.

If two circles intersect in P, Q the angle between the tangent at P, Q is equal to the angles which the centres subtend at S and supplementary to the angle which PQ subtends at the other point of intersection.

38 and 39. If from any point P in the radical axis tangents be drawn to the circles, and a circle be described, with centre P and radius equal to the tangent, this circle will intersect the line of centres in two points E and F which are the limiting points of the system.

Take A at centre of one of the circles, and M at the point where the radical axis intersects the line of centres.

Then, PU and PU' being the tangents from P to the circle,

$$PM^2 + ME^2 = PE^2 = PU^2 = PA^2 - AU^2;$$

$$\therefore\ ME^2 = AM^2 - AU^2 = MF^2;$$

$$\therefore\ AE \,.\, AF = AM^2 - EM^2 = AU^2.$$

$\therefore$ the polar of F passes through E.

Reciprocating with regard to F, the pole of uU', i.e. of the fixed line through E, is the centre, which is therefore fixed, and the conics are confocal.

Therefore, if we reciprocate with regard to either limiting point we obtain confocal conics.

40. If perpendiculars be drawn from A, B, C to BC, CA, AB these lines will meet in a point O, and the circles circumscribing ABC, OBC, OCA, OAB are equal.

41. If the tangents at P and Q, points on a circle intersect at a constant angle, and lines be drawn through

P and Q making constant angles with the tangents at P and Q respectively, this pair of straight lines will intersect on a concentric circle.

42. If two circles intersect in A and B and PQ be a common tangent and QB, PA meet the circles in C and D, then PC, QD are parallel.

43. If from any point on a circle circumscribing a triangle perpendiculars be drawn to the sides of the triangle, the feet of these perpendiculars lie on a straight line.

44. Since the orthocentre is on the hyperbola, DEF is a self-conjugate triangle and the pole of EF lies on BC.

Hence the pole of BC lies on EF.

45. If AB, CD meet in the fixed point E, CA and BD in F, and BC and AD in G, then FG is the polar of E.

Hence the centre of the circle lies in a straight line through E perpendicular to FG the polar of E with respect to both curves.

46. The radius of an escribed circle of an equilateral triangle is $\frac{3}{2}$ the radius of the circumscribed circle, and if SE be the tangent from the centre of the circumscribed circle to the escribed circle whose centre is D;

$$SD = DE + \tfrac{1}{3}DE = \tfrac{4}{3}DE.$$

The proposition in the question is obtained by reciprocating with respect to the circumscribed circle.

47. If AD be drawn parallel to the axis to meet BC, AD is bisected at D' where it meets the curve.

Hence the tangent at D' is parallel to BC and bisects AB and AC.

Since a straight line intersects a conic in two points and two tangents can be drawn from a point, the reciprocal polar of a conic with respect to another conic is a third conic.

Now by Ex. 44, if a rectangular hyperbola circumscribe a triangle *DEF* it will go through the ortho-centre *O* and *ABC* the triangle formed by joining the feet of the perpendiculars is a self-conjugate triangle, and *O* is the centre of the circle inscribed in *ABC*. If we reciprocate with respect to *O* the reciprocal conic is a parabola, since it has one tangent at an infinite distance and *ABC* is a self-conjugate triangle.

The tangents from *O* are at right angles, since the hyperbola was rectangular, hence *O* is on the directrix.

The locus of the poles of the lines at an infinite distance, that is, of the centres of the hyperbolas, was the circle circumscribing *ABC*.

Hence the envelope of the polars of *O* with respect to the parabolas is an ellipse inscribed in *ABC* having *O* for a focus. Since *O* is now the centre of the circle circumscribing *ABC*, the auxiliary circle of the ellipse is the nine point circle.

MISCELLANEOUS PROBLEMS.

1. If S and H be the rifle and target, and P the hearer, the difference of the times in which sound travels from S and H to P is equal to the time of the bullet's transit from S to H.

Hence $HP-SP$ is constant, and the locus is a hyperbola of which S and H are the foci.

2. Let tp, tq be the tangents parallel to PQ and $P'Q'$ and let qt meet in r the diameter through p; then $qt=tr$, and

$$\begin{aligned} PR^2 : PQ^2 &:: tr^2 : tp^2 :: tq^2 : tp^2 \\ &:: SP'.SQ' : SP.SQ \\ &:: P'Q' : PQ; \qquad \text{Art. 17.} \\ \therefore PR^2 &= PQ.P'Q'. \end{aligned}$$

3. $$QN : CM :: BC : AC :: DM : CN,$$

hence $$QN+DM : NM :: BC : AC.$$

4. If $CVED$ be conjugate to PQ,

$$PQ.Qp=PV^2-QV^2.$$

Hence $PQ.Qp : CD^2-CE^2 :: CR^2 : CD^2$, CR being parallel to PQ.

5. $$QN : NX :: SA : AX :: SR : SX.$$

Hence Q lies on the tangent at the extremity of the latus rectum.

6. $$PN^2 : AN.NA' :: BC^2 : AC^2$$

and $$QN.PN=AN.NA'.$$

Therefore $$QN^2 : AN.NA' :: AC^2 : BC^2.$$

7. Let AP, QB meet in R, and draw RV parallel to POQ.

Then $$RV : VA :: PO : AO,$$

and $$RV : VB = QO : OB.$$

Hence $$RV^2 = VA \, . \, VB,$$

since $$AO \, . \, OB = PO \, . \, OQ.$$

Therefore QR lies on a concentric rectangular hyperbola.

8. The line joining T to the intersection of the normals at P and P' bisects PP' and therefore passes through the centre.

9. If the tangent at P meet the tangents at A and A' in T and T' and TS, $T'S'$ meet in Q,

the angle $$SS'Q = T'S'A' = T'S'P;$$

$$QSS' = AST = TSP;$$

and $$SPT = S'PT'.$$

Hence S, P, S' are the feet of the perpendiculars of the triangle TQT'.

Therefore QP is perpendicular to TP.

10. Make the angle PSF a right angle, then the tangent at P meets the directrix in F: if a circle be described with centre P and radius PK such that the ratio $SP : PK$ is equal to the eccentricity the directrix is a tangent from F to this circle.

Two tangents can in general be drawn.

If the angle SPF be such that $SP : PF :: SA : AX$, only one conic can be constructed; there are two positions of PF equally inclined to SP corresponding to this case.

If the eccentricity be unity, one conic is a line parabola through S.

11. If PK be drawn perpendicular to the directrix of the parabola $SP = PK$,

hence $$HM = HP + PK = AA'.$$

Therefore the directrix touches a circle of which H is centre.

12. Draw RN perpendicular to the minor axis.
Then

$$CN.AC=SR.AC=BC^2=AC^2-SC^2=AC^2-RN^2;$$

Hence $$RN^2=AC\{AC-CN\},$$

or R lies on a parabola.

13. If ST, PQ meet in H, $HTRS$ is a harmonic range.

But OH, OT, OV, OS is a harmonic pencil.

Hence OV passes through R.

14. Let ACA' be the diameter bisecting the parallel chords QN, etc. in N, etc.

Then PN^2 varies as QN^2, that is as $AN.NA'$.

Hence the locus of P is an ellipse. The locus will be a circle if $PN=QN$, that is if the vertical angle is a right angle.

15. If PP', QQ' be the double ordinates of the given points, P, P', Q, Q' are fixed points, and since the ellipses are similar, the corresponding points of the auxiliary circle, at which the major axis subtends a right angle, are likewise fixed points.

16. The ordinates of the point and of the end of one of the radii are in the ratio of the radii.

17. If P be the centre of the circle and PK perpendicular to the fixed straight line, the ratio $SP : PK$ is constant.

18. Let pqr be a triangle touching the parabola in P, Q, R.

Parabolic area $PQR=\frac{2}{3}$ triangle PqR.

$$\therefore \text{Triangle } PQR=\tfrac{2}{3}(PqR-PrQ-QpR),$$

$$3PQR=2(pqr+PQR);$$

$$\therefore PQR=2pqr.$$

19. If a rectangular hyperbola circumscribe a triangle the orthocentre is on the curve, if we reciprocate with respect to the orthocentre we have the case of a parabola inscribed in a triangle the tangents from the orthocentre being at right angles. See Ex. 47, Ch. 10.

20. Produce HA to K making AK equal to AH, then PK and AL are parallel.

Hence $SQ : SP :: SA : SK :: SA : SA + AH.$

Therefore the locus of Q is a similar ellipse of which S is focus.

21. If $KLMN$ be the quadrilateral and k, l, m, n the points of contact, KM will bisect nk, lm: and LN will bisect kl, mn.

Hence $klmn$, and therefore $KLMN$, is a parallelogram.

22. The locus of the second focus is a circle of which the radius $= AA - SP$.

The locus of the centre which bisects SH is similar, that is, a circle.

23. Since LC, LL' are tangents, the angles HLC and SLL' are equal.

Again $\qquad CL \,.\, CL' = CH^2,$

hence the angle $\quad CHL = CL'H = SL'L.$

Therefore $\qquad CL : HL :: SL : LL',$

the triangles CLH, SLL' being similar.

24. If the theorem be true in the case of a circle, it will follow by orthogonal projection for any ellipse.

If PQ be the chord of contact of tangents drawn to a circle from a point on a concentric circle, the angles PAQ, $PA'Q$ will be constant, A, A' being extremities of a fixed diameter.

Let AP, $A'Q$ meet in R, and $A'P$ and AQ in R'.

The angle $\qquad ARA' = APA' - PA'Q,$

and the angle $\quad AR'A' = APA' + PAQ.$

Hence the loci of R and R' are circles passing through A and A'.

25. Circumscribe circles to two of the triangles formed by the intersections of the tangents, these circles intersect in the focus S: the pedal line of S is the tangent at the vertex.

A parabola can be drawn to touch five straight lines, if the circles circumscribing the triangles formed as above all meet in the same point S.

26. PF is the same for both curves, and therefore CD is also the same.

27. Prove that $SY \,.\, S'Y'$ is constant.

28. By reciprocation.

29. P, Q, R, R' lie on a circle of which PQ is diameter and PQ, LL' are equally inclined to the axis. If p, p' are the vertices of the diameters bisecting PQ, RR' in V and V', pp' is a double ordinate.

Let VV' which is parallel to the normal at p' meet the axis in O.

Let VM, $V'M'$ be the ordinates of V and V'.

Then $LL' = L'M' + M'O + MO - LM = 2OM = 2ng = 4AS$.

30. The bisectors are tangent and normal to a confocal conic.

Hence $$CG \,.\, CT = CS^2.$$

31. Reciprocate the following theorem: if S, A, B, C be points on a circle and with centres A, B, C and radii AS, BS, CS circles are described, they will intersect two by two in points which lie in a straight line.

32. $OE : EG = SP : SG = Sp : Sg = OE : Eg$.

33. If an ellipse be reciprocated with respect to its centre, the reciprocal is a similar ellipse having its major axis in the minor axis of the original ellipse.

If we reciprocate an ellipse circumscribing a triangle and having its centre at the orthocentre with respect to that orthocentre, the reciprocal is a similar ellipse, inscribed in a triangle having its sides parallel to those of the original triangle, the homologous axes being at right angles, and having its centre at the orthocentre.

This reciprocal ellipse is similar and similarly situated to the ellipse inscribed in the original triangle having its centre at the orthocentre.

34. Q lies on the common circle of curvature,

hence $$PQ = 4PT.$$

35. AB, BC are equally inclined to the axis, hence since the angles at A and C are equal, AD, DC are equally inclined to the axis.

Hence the tangent at D and AC are equally inclined to the axis.

Therefore the tangents at B and D are parallel.

36. The volume cut off varies as the area VAA' and BB'; and the area VAA' varies as $AV \, . \, VA'$ or $AD \, . \, A'D'$, that is BC^2.

37. $$SQ : Pg :: St : tg :: SY : SP :: BC : CD :: PF : AC.$$

Hence $$SQ \, . \, AC = PF \, . \, Pg = AC^2,$$

or $$AC = SQ.$$

Let QL be the ordinate of Q and let MQ meet the major axis in V.

Then $$CV : PN :: CV : CM :: CL : CM - QL :: CL : PN - QL,$$

and $$SP - AC : CN :: SC : AC,$$

or $$AC \, . \, SP = AC^2 + CN \, . \, SC = BC^2 + CS \, . \, SN.$$

Again $$CN - CL : SP - SQ :: SN : SP,$$

and $$SP - SQ : CN :: SC : AC;$$

hence $CN-CL : CN :: SN.SC : SP.AC;$

therefore $CL : CN :: BC^2 : SP.AC,$

or $CL : SP-SQ :: BC^2 : SP.SC.$

Now $SP-SQ : PN-QL :: SP : PN,$

hence $CL : PN-QL :: BC^2 : PN.SC.$

Hence $$CV.SC=BC^2,$$

or V is a fixed point.

38. If SL, SM, SN be drawn perpendicular to the given tangents, the circle circumscribing LMN is the auxiliary circle of the conic.

39. If S be the centre of the circumscribed circle, H the orthocentre, the centre of the nine-point circle bisects SH: and if PQR be the triangle, the angles SPQ, HPR are equal: hence S and H are the foci.

40. If Pg, $P'g'$ be the normals at P and P' and gL, $g'L'$ be drawn perpendicular to PP',

then $PL=P'L'$, by Ex. 27, Chapter I.;

hence $$gG=Gg'.$$

Therefore

$$2SG : SP+SP' :: Sg+Sg' : SP+SP' :: SA : AX.$$

41. Reciprocate the following with respect to S: ASB is a diameter of circle meeting a concentric circle in S, the opposite sides of the quadrilateral formed by tangents through A and B to the inner circle are parallel, and the tangents to the outer circle at the points where it meets the tangent at S are respectively parallel to them.

42. If P, Q be two points on a rod and PS, QS are at right angles to the directions of motion of P and Q, then if R be any point on the rod the direction of motion of R is at right angles to SR.

Hence the directions of motion of all points on the rod envelope a parabola of which S is focus and the rod tangent at the vertex.

43. SP and HQ are parallel to CT.
Hence PSp = complement of SPp = complement of CTP.
QHq = complement of HQq = complement of CTQ.
Hence PSp and QHq are together equal to the supplement of PTQ.

44. Let ST, $S'T$ meet CP in p and p', and CD in d and d'.

Since the angle $\qquad PTS = d'TD$,

and $\qquad TPp = TDd'$,

the angles SpC, $Cd'S'$ are equal, and d, d', p and p' lie on a circle.

45. If the asymptote and directrix meet in D, SDC is a right angle and if DP be the tangent PSD is a right angle.
Therefore SP is parallel to the asymptote.

46. If PTQ, ptq be two consecutive positions and TV, ty be drawn perpendicular to pt, TQ respectively,

$$tV = Pp + pt - PT = PT + TQ + Qq - tq - PT$$
$$= TQ + Qq - tq = Ty.$$

Hence the tangent at T is equally inclined to PT and TQ.
Hence T lies on a confocal ellipse.

47. CP and CQ are at right angles; hence C, P, D, Q lie on a circle.

48. If PV be the chord through the centre, and pp' the parallel focal chord,

$$PV \, . \, CP = 2CD^2,$$

and $\qquad pp' \, . \, CA = 2CD^2$.

Hence $\qquad PV : pp' :: CA : CP$.

49. SP and QH are both parallel to CT;
hence the angle

$$pCq = pCT + TCq = PHQ + QSP.$$

50. The angle SPY = half the supplement of SPS' = half PSP'.

51. QSP, $Q'SP'$ are right angles and the perpendiculars from Q, Q' on SP, SP' are equal to the perpendiculars on PP'.
Hence QP, $Q'P'$ are tangents at Q and Q' to the parabola of which S is focus and PP' directrix.
And the diameter parallel to PP' is tangent at the vertex, since it bisects SH. BB' is a tangent since SCB is a right angle.

52. SE will evidently envelope a conic of which S is focus and the given circle auxiliary circle.

53. Join SP, the bisector of POS bisects SP in V; since the locus of P is a circle, the locus of V is a circle.
Hence VO envelopes a conic of which S is a focus.

54. The angles RSP and QHV are the complements of PST and QHT.
Hence $RSP + QHV =$ supplement of half $PSQ + PHQ$ $=$ half the angle between the tangents at P and Q, by Ex. 23 on the ellipse.

55. TS, ZS are the interior and exterior bisectors of the angle QSR,
and if TQ meet PZ in F, FS is the exterior bisector of the angle QST.

Hence Q, T, R lie on a conic of which S is focus and PZ directrix and ZT is the tangent at T since ZST is a right angle.

56. If OE be the radius of the sphere,
$OE \cdot AC =$ area OAA'
and varies as $OA \cdot OA'$ or $AD \cdot A'D'$ that is as BC^2.
Hence the latera recta of all the sections are the same.

57. Let PQ', QP' meet the ellipse in U and V,
then since $TP^2 = TQ \cdot TQ'$
and $TQ^2 = TP \cdot TP'$,

$$TP : TP' :: TQ' : TQ,$$

or PQ', $P'Q$ are parallel.

Hence if CF be parallel to $P'Q$ and PQ',

$$P'V.P'Q : P'P^2 :: CF^2 : CD^2,$$

and $$Q'U.Q'P : Q'Q^2 :: CF^2 : CE^2.$$

Now $$P'P^2 : Q'Q^2 :: TP^2 : TQ'^2 :: TQ : TQ',$$

and $$CE^2 : CD^2 :: TQ^2 : TP^2 :: TQ : TQ'.$$

Hence $P'V.P'Q : Q'U.Q'P :: TQ^2 : TQ'^2 :: P'Q^2 : PQ'^2$.

Therefore $$P'V : P'Q :: Q'U : Q'P.$$

58. If the tangents at P and Q meet in T, CT which bisects PQ is parallel to $P'Q$, P' being the other extremity of the diameter PCP'.

Hence the angle $PCT = PP'Q = TPQ$.

59. S, P, S', g lie on a circle,

hence $$Sg : Pg :: S'G : S'P :: SA : AX.$$

60. BC is parallel to the polar of A, hence AD is parallel to the axis.

Also the angles SAC, DAB are equal: and the angles ABD, ASC are likewise equal.

Therefore $$AC : AS :: AD : AB.$$

61. $$RK : QN :: KC : NC$$

and $$PN : RK :: NG : KG.$$

Hence $$BC : AC :: KC.NG : NC.KG,$$

or $$KC : KG :: AC : BC.$$

Therefore $CR : CQ :: CK : CN :: AC + BC : AC$,

or $$CR = AC + BC.$$

Hence if NP meet RL in N', $PN = QN$.

Hence KL passes through P.

Also $$PL = QC = AC,$$

and $$KP = QR = BC.$$

62. $$CT.CN = CA^2$$

and $$CT'.PN = BC^2;$$

hence $CT.CT' : CA.CB :: CA.CB : CN.PN.$
Hence the triangle CTT' varies inversely as the triangle PCN.

63. By Ex. 6. Chap. VII.

$$2SP = 3AC,$$

and if CE be drawn parallel to the tangent at P,

$$PE = AC.$$

64. Let CT which bisects PP' in V, meet the ellipse in Q, and let CE be conjugate to CQ.

Then $PV^2 : CV.VT :: CE^2 : CQ^2 :: PV^2 : CQ^2 - CV^2$.

Hence $\qquad CQ^2 = CV.VT + CV^2 = CV.CT$,

or TP, TP' are tangents.

65. See Ex. 82 on the hyperbola.

66. If we reciprocate with respect to a focus the theorem that tangents to an ellipse at right angles intersect on a fixed circle, we find that if the sides of a quadrilateral $ABCD$ subtend each a right angle at a fixed point S the sides envelope an ellipse of which S is a focus. If O be the centre,

the angle OAB = complement of half AOB = complement of $SCB = CBS = SAD$.

Hence O is the other focus.

67. If A', B', C', D' be the points of contact and E', F', G' the points of intersection of $A'C'$, $B'D'$; $A'D'$, $B'C'$; $A'B'$, $C'D'$,

then $E'F'G'$ is a self-conjugate triangle.

If $BA'A$, $C'DF$ meet in F the pole of $A'C'$, F will lie on $F'G'$ the polar of E', since E' lies on $A'C'$ the polar of F.

Similarly $AD'D$, $BB'C$ meet in G on $F'G'$ and AC, BD in E'.

Let $A'D'$, $CC'D$ meet in α, and $B'C'$, $AD'D$ in β.

Then if O be intersection of AC', CD', $\alpha\beta$ is the polar of O.

Now since GB', GD' are tangents and the tangent at C' meets them, the range $CC'DF$ is harmonic and therefore the pencil GB', GC', $G\beta$, GF'.

Hence $B'C'\beta F'$ is a harmonic range.

So $A'D'\alpha H$ is a harmonic range.

Hence $\alpha\beta$ passes through G'.

Since G' lies on the polar O, O lies on BD, the polar of G.

Similarly AB', CA' intersect on BD.

68. By Art. 138 the four points in which two rectangular hyperbolas intersect are such that any one of them is the orthocentre of the triangle formed by the other three: hence any conic through the four points is a rectangular hyperbola.

69. If KVt be drawn parallel to the axis to meet, PQ, ST in V and t, and if KL be perpendicular to PQ, $PL=SQ$ and $PV=VQ$, hence $tV=VK$. Therefore ST SP, SK and the axis form a harmonic pencil.

70. Let DE meet PP' in K; and let PD, $P'E$ meet in H.

Then GH is the polar of K, but K lies on DE the polar of F.

Hence F lies on GH, and FG is parallel to the chords bisected by PP'.

71. Let RR' the common tangent be bisected by PQ the common chord in O.

Then $$RO^2=OR'^2=OQ\,.\,OP\,;$$

hence RR', PQ are equally inclined to the axis.

Hence PR is a diameter, and the diameter of curvature $=2PF=2CD$.

Therefore $$CD^2=CD\,.\,PF=AC\,.\,BC.$$

72. Reciprocate with respect to S the following theorem: S is taken on the outer of two concentric circles; SY, SZ are drawn perpendicular to a pair of parallel tangents to the two circles; YZ is constant.

73. Let the tangents at P and Q meet in T.
Then the angle $P'RQ' = PTQ =$ supplement of PCQ, by Ex. 58.
Hence P', C, Q' and R lie on a circle.

74. TN is half the difference of QM and $Q'M'$ and RR' is half their sum.

Hence $$R'P = \tfrac{1}{2}\, Q'M' = KR.$$

Now $PN : Q'M' :: TK : 2KR :: PM : QM :: QM : 4SP$.

Hence $$PN : QM :: Q'M' : 4SP :: PM' : Q'M'.$$

Therefore $$PN^2 : PM'^2 :: QM^2 : Q'M'^2 :: PM : PM',$$

or $$PN^2 = PM \,.\, PM'.$$

75. Let $RR'V$ be the diameter bisecting PQ.
Then if PQ meet a common tangent pp' in O,
$$Op^2 : OP\,.\,OQ :: Sp : SR :: S'p' : S'R' :: Op'^2 : OP\,.\,OQ,$$
or $$Op = Op'.$$

76. Let O, C be the centres of the hyperbola and ellipse to $tbCb't'$ the tangent at C, then since $PN\,.\,Ct = Cb^2$ and $CN\,.\,CO = Ca^2$, the area of the ellipse will be a maximum, when $CN\,.\,PN$ is maximum, that is, when $CN\,.\,NO$ is a maximum, or $ON = NC$.
Hence PP' is a tangent to a similar hyperbola.

77. $RP\,.\,RP' = RN^2 - PN^2 = RN^2 - 4AS\,.\,AN$.

Now $$RN^2 : 4AS\,.\,AA' :: AN^2 : AA'^2,$$

or $$RN^2 : 4AS\,.\,AN :: AN : AA'.$$

Hence $$RN^2 - PN^2 : 4AS\,.\,AN :: A'N : AA'.$$

Therefore $$RP\,.\,RP' : AN\,.\,A'N :: 4AS : AA'.$$

78. Let the tangent at P meet the confocal conic in Q.
Draw CEF parallel to PQ meeting the normal at P in F.

Then $$OP\,.\,PF = CD^2 = SP\,.\,PS' = Cd^2,$$

Cd being conjugate to CP.
Hence O is the pole of PQ with respect to the confocal.

79. Let the tangents meet in U, SU meeting the curve in Q, and let the tangent at Q meet RR' in T. Then, V being point of contact of RR',

$$TSR = TSQ + USP - RSP = TSV + USP' - RSV$$
$$= USP' - RST,$$
$$\therefore\ 2TSR = USP' = RSR';$$

$\therefore$ locus of T is tangent at Q.

Or by reciprocation of the theorem,

If ABC be a triangle inscribed in a circle, and DE the diameter perpendicular to AC, DB and EB bisect the angle B and its supplement.

80. Reciprocate with respect to any point S the theorem that if two points on a circle be given, the pole of PQ with respect to that circle lies on the line bisecting PQ at right angles.

81. $\qquad PQ.PR = PE.PF = AC.BC$

and $\qquad QR = CE = AC - BC.$

Hence $\qquad PQ = BC$ and $PR = AC.$

Also $\qquad ER$ is parallel to CQG.

Hence $\qquad PG : CD :: PG : PE :: BC : AC$

and $\qquad PG.PF = BC^2.$

Hence CQ, CR are the axes.

82. This is a particular case of Art. 195, since the second point where AE meets the curve is at an infinite distance, hence $AE = EK$.

83. The circle of curvature is greatest at the extremity of the minor axis.

Hence BO the direction of the minor axis is given.

And $BC.BO = AC^2 = SB^2$, O being the centre of curvature.

Hence S lies on the circle of which BO is diameter.

84. $\qquad Ca : Cb :: Ba.Ac : bA.cB$

and $\qquad Ca : Cb' :: Ba.Ac' : b'A.c'B,$

by Todhunter's Euclid, Art. 59.

And $Ac.Ac' : Ab.Ab'$ in ratio of squares on parallel diameters.

Hence BC is the tangent at a.

85. This depends on the fact that any chord is bisected by the diameter through the intersection of the tangents at the ends of the chord.

86. Let P, Q be the points of contact of parallel tangents to the conic and circle.

Then by Art. 132, the angles PCA, QCA are equal.

87. Let CL, CL' be the fixed straight lines, S the fixed point.

Then the angle $LSL' = CSL' + CL'S$,

hence the angles $CL'S$, CSL are equal.

Therefore $CL : CS :: CS : CL'$,

or LL' touches the hyperbola of which CL, CL' are asymptotes and S focus.

88. Since the semivertical angles are complementary they touch one another along their common generating line.

Now $EA : AX :: CE : EO :: OE' : C'E' :: EA : AS'$.

Hence S' coincides with X, and similarly S with X'.

89. Draw CRV perpendicular to the tangent at P, CE bisecting PP' and PN parallel to CE.

Then $QO.OQ' : PO.OP' :: CD^2 : CR^2 :: PO.PF : CN.CV :: PO : PE :: 2PO : PP'$.

90. A circle can be described with centre T to touch SP, SQ, HP, HQ.

Hence $$SN - NH = SM - MH,$$

and TM, TN bisect the angles at M, N.

Hence TM, TN touch a confocal conic passing through M and N.

91. If S and H are the given points, the locus of P is the conic in which the given plane through S intersects the

surface generated by the revolution about SH of a conic of which S and H are foci.

If ST be drawn perpendicular to the plane to meet the directrix plane corresponding to S in T, the cone formed by joining H to all points of the locus of P is a right circular cone of which TH is axis.

92. $$PG \, . \, Pq = BC^2 = PQ^2;$$

$\therefore$ PGQ and PgR are similar triangles.

93. If OL be the ordinate of O,

$$LG : GN :: OL : P'N :: CL : CN,$$

or $$LC : LG :: CN : NG :: AC : BC^2.$$

Hence $$CL : CG :: AC^2 : AC^2 + BC^2.$$

Therefore $CO : CP' :: CL : CN :: AC^2 - BC^2 : AC^2 + BC^2$.

94. The polygons $V'SPV$ and $Z'HPZ$ are similar and the perpendicular from C on VZ bisects VZ;

hence if VE be taken on VV' such that $VE = ZZ'$,

then $$CE = CV' = CZ'.$$

Hence $VV' . ZZ' = VV' . VE = VC^2 - V'C^2 = CA^2 - CA'^2$.

95. The centre is the middle point of CP.

96. TSQ and $TS'Q$ are right angles;

$\therefore$ the middle point of TQ is the centre of the circle $TSQS'$ and is equidistant from S and S'.

97. $$TQ : TP :: SQ : ST,$$

and $$T'P : T'Q' :: ST' : SQ';$$

$$\therefore TQ . T'P : TP . T'Q' :: SQ . ST' : SQ' . ST$$
$$:: SQ . PT' : SQ' . PT.$$

98. Draw NE perpendicular to NM, and prove that E is a fixed point in the axis.

99. P and Q are equidistant from the plane of the circular section of the cone, which contains the centre of the section.

100. Produce OC to E so that $CE = OC$;

then PE is parallel to CZ and $EPY = CZO = OPY$; that is, the tangent to the curve bisects the angle OPE.

101. If PEQ be one of the tangents, and ERV the chord,

$$EP^2 : ER . EV :: EQ^2 : ER . EV,$$

for each ratio is that of the parallel focal chords.

102. If FE, FG be the tangents, $F(TEQG)$ is harmonic, and EFG is a right angle.

103. Reciprocate with regard to C the theorem, that, if a circle centre C intersect another circle at right angles at the point E, and CPQ be any chord, $CE^2 = CP . CQ$.

104. Reciprocate the conics into two intersecting circles.

105. Reciprocates into the theorem of the existence of the director circle.

106. If PEQ be the chord required, and $P'EQ'$ a consecutive chord, the areas PEP', QEQ' are ultimately equal, and E, which is the centre of curvature, is the middle point of PQ.

PQ is therefore the diameter of curvature and is inclined to the axis at the same angle as the tangent, i.e. half a right angle.

107. The pole F of the straight line is fixed, and P, the point of contact of a tangent, is the foot of the perpendicular from F on the normal.

108. The angle $MNC = LCN = LCN$, if l be the point where the tangent meets the other asymptote.

$\therefore MN$ is parallel to Cl, and passes through P the middle point of Ll.

109. The diameter of curvature being the same for both, it follows that $SP : S'P$ is a constant ratio.

110.

$$CK'^2 = CP^2 + PK'^2 + 2PK' . PF = AC^2 + BC^2 + 2AC . BC;$$

$$\therefore CK' = AC + BC.$$

111. If P, V, R, Q be the points of contact of AB, BC, CD, DA,

$$2ASB = PSV + PSQ,$$

and two right angles

$$= PSV + ASP + VSC = PSV + ASQ + CSR;$$

$$\therefore PSV = QSR = 2QSD,$$

and $$2ASB = 2QSD + PSQ = 2ASD.$$

$$\therefore ASB = ASD = \text{a right angle.}$$

Also $$ASP + DSR = ASD,$$

$\therefore PSQ$ is a straight line and PA, RD intersect on the directrix.

112. If the tangent at P meet the director circle in R and T, perpendiculars to the tangent through R and T are tangents to the ellipse.

Draw PE parallel to CR, meeting CT in V and take $CQ^2 = CV \cdot CT$; similarly find the point D on CR;

then CQ and CD are conjugate diameters, and the construction is completed in Art. 216.

113. $Q'R'$ meets the axis in T, the pole of NP;

$\therefore$ the tangents at Q', R', meet at a point E on NP.

Let CE meet $Q'R'$ in Y;

then $EN \cdot PN = EN^2 - EN \cdot EP = EC^2 - CN^2 - CY \cdot CE,$

$$= EC \cdot CY - CN^2 = CA^2 - CN^2 = P'N^2.$$

$$\therefore EN : P'N = P'N : PN = AC : BC = QM : QM,$$

and the tangent at Q passes through P'.

114. If S' be the other focus of the fixed ellipse, and H of the moving ellipse,

$$S'P = HP \text{ and } S'Q = HQ.$$

Join $S'H$ meeting the chord in Z, and let fall SY the perpendicular on the chord;

then $SY \cdot S'Z = SY \cdot HZ = BC^2$, and in the chord touches a confocal conic.

115. Let O be the centre of the circle, PQ a chord of intersection not perpendicular to the axis, meeting an asymptote in L, and the axis in K.

$$\text{angle } EOC = 90^\circ - LKC = 90^\circ - LCA + CLK$$
$$= 45^\circ + CLK = LCA + LCE = ECO,$$

$\therefore$ ECO is isosceles, and E lies on a fixed line perpendicular to the axis.

116. Project the ellipse into a circle, and prove that the angle $DQP = QPR$, observing that the tangent and common chord are equally inclined to the axis.

117. Reiprocates into the following:

If S be a fixed point, and SK a tangent to a circle centre C, and if TE be any other tangent from a point T, and the angle $CTE = CSK$, the locus of T is a circle passing through S.

118. SY, HZ being perpendiculars on the tangent,

$$Pq : HZ :: PT : TZ :: TR : TC :: PR : PE;$$
$$\therefore Pq : PR :: HZ : AC :: HP \,.\, BC : AC \,.\, CD$$
$$:: HP \,.\, PG : CD^2 :: PG : S'P;$$

$\therefore$ Rq is parallel to SG.

119. If PT, QT and pt, qt be two near positions and tM, Tm be drawn perpendicular to PT, Qt,

then $tM : Tm$ in the ratio compounded of $PT : QT$ or $CD : CE$ and $Pp \,.\, QO' : Qq \,.\, PO$ or $PF : QF' : QO'$, PO being the radii of curvature.

Hence $tM = Tm$, or the normal at T to the locus of T bisects PTQ.

Therefore T lies on a confocal ellipse.

120. Referring to the figure of Art. 148, and drawing the lines, a circle can be drawn through $ADOG$;

$$\therefore \text{angle } DOA = DGA = 90^\circ - GVA = AEC,$$

and the triangles AOD, ACE are similar.

$$\therefore AO : AD :: CE : AC,$$

or $$AO \,.\, AC = AD \,.\, A'D' = BC^2.$$

121. Referring to the preceding theorem, describe a sphere centre G and radius GA; the tangent from any point of the ellipse to this sphere will be equal to the tangent from P to the circle of curvature.

Describing a similar sphere with centre G', the sum of the tangents $= FA' = SS'$.

122. In the second figure of Art. 144, take T any point in the tangent at P and let C be the centre of the upper sphere.

Then CRP and CSP are right angles, $PR = PS$, and $TR = TS$ these lines being tangents.

$\therefore$ T lies in the plane through CP perpendicular to the plane $CRPS$;

$$\therefore \text{angle } SPT = RPT, \text{ and } RTP = STP;$$

similarly $$R'TP = S'TP.$$

Hence $RTR' = STP + S'TP = STP + STQ = PTQ$,

TQ being the other tangent.

123. Produce ER to E' making RE' equal to RE.

Then the polar of E' passes through E.

Now C is the pole of PQ which passes through E.

Hence CE' is the polar of E and is therefore parallel to ARB.

Hence CE is bisected by AB.

Again CE bisects in E the polar of E' which is parallel to AB.

Therefore CE bisects AB.

Therefore $ACBE$ is a parallelogram.

124. Let O be the centre of the conic which touches the sides AB, BC, CD, DA in E, F, G, H.

Then since OA, OB, OC, OD bisect HE, EF, FG, GH the sum of the areas of the triangles AOB, COD is half the area of the quadrilateral.

Let AB, CD meet in K and let O, O' be two positions of O.

Draw OM, $O'M'$ perpendicular to AB and ON, $O'N'$ perpendicular to CD.

Also draw $O'K$, $O'L$ perpendicular to OM and ON:

then $OM \cdot AB + ON \cdot CD = O'M' \cdot AB + O'N' \cdot CD$.

Hence $$OK \cdot AB = OL \cdot CD.$$

Therefore $OL : OK$ in a constant ratio.

Hence the locus is a straight line.

125. By Art. 241 the sum or difference of the tangents is proportional to the distance between the ordinates of the points where the circles touch the curve according as the point does or does not lie between those ordinates.

126. If SY, $S'Y'$ be the perpendiculars from the focus on the tangent at P, CY' is parallel to SP; and, if DK is the perpendicular on CY',

$$DK : CD = SY : SP = BC :: CD;$$
$$\therefore DK = BC.$$

127. If V and T are contiguous corners of the parallelogram formed by the tangents, and if CV and CT meet in E and F the sides of the parallelogram formed by the points of contact,

$$CE \cdot CV = CP^2 \text{ and } CF \cdot CT = CD^2;$$
$$\therefore (CE \cdot CF)(CV \cdot CT) = (CP \cdot CD)^2.$$

128. Taking the figure of Art. 12, let TL, TM, TN be drawn perpendicular to SF, SP, FF', and let the circle intersect FF' in G and G';

then $$TG : TN :: TL : TN :: SM : TN$$
$$:: SA : AX,$$
$$\therefore TG \text{ is parallel to an asymptote.}$$

129. For TS bisects QSq, Art. 12, and FS bisects the outer angle, Art. 5.

130. Let the chord Qq, normal at q, meet the directrix in F, and let T be the pole of Qq; then S being the pole of the directrix, ST is the polar of F, and therefore T being a point in the directrix, TSF is a right angle.

Taking V as the middle point of Qq, let FS meet in L the polar of V, which is parallel to Qq.

Then $LSQ = TSQ - TSL = TSq - TSF$
$= FSq = SqV - SFq$
$= PqV - STq$, $\because$ T, S, q, V are concyclic,
$= PqV - QTV$, Art. 39,
$= PVq - QTV = QVN - QTV$
$= TQV = LTQ$, $\because$ TL, QV are parallel.

$\therefore$ L, T, S, Q are concyclic, and
$TQL = TSL = 90^0$.

131. (1) If the plane through the axis and the given point P intersects the cone in VA, VB, describe a circle passing through P and touching VA, VB; then if APB is drawn touching the circle, AB is the axis of the section of which P is a focus.

(2) Produce VP to Q making $PQ = PV$, and in the plane above mentioned, draw VK parallel to VB, meeting VA in K, and VL parallel to VA, meeting VB in L;

Then APL is the axis of a conic of which P is the centre.

132. If S, S' are the foci, and X the foot of the directrix, $VS'S$ is a straight line, and XSS' is an isosceles triangle.

Taking A and A' as the corresponding vertices, draw AL and $A'L'$ parallel to SS', meeting AS' in L and AS in L'.

The latera recta are in the ratio of VS to VS',
and $VS : AS = A'L' : AL'$ and $VS' : A'S' = AL : A'L$.

Now $AX = LX$, $A'X = L'X$, and $A'L = AL'$;
but $VS . AL' = AS . A'L'$ and $VS' . A'L = A'S' . AL$;
$\therefore$ $VS : VS' = AS . A'X : A'S'AX$.

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